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The 2-Blocks of the Covering Groups of the Symmetric Groups

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Let $\hat{S}_n$ be a double cover of the finite symmetric group S_n of degree n, i.e., $\hat{S}_n$ has a central involution z such that $\hat{S}_n/\langle z\rangle \simeq S_n$. An irreducible character of $\hat{S}_n$ is called *ordinary* or *spin* according to whether it has z in its kernel or not.

The purpose of this paper is to determine the distribution of the spin characters of $\hat{S}_n$ into 2-blocks. The methods applied here are essentially different from those applied to previous questions of this type. We also discuss some consequences of our main result for the decomposition numbers. An analogue of James' well-known result for the decomposition numbers of the symmetric groups is proved, providing also a generalization of a theorem of Benson [Ben, Theorem 1.2]. In Section 1 we present the background for our results and give some preliminaries. In Section 2 we give an explicit formula for the number of spin characters in a 2-block. We also prove a result about the weight of a block containing a given non-self-associate spin character which will be important for the proof of our theorem on the 2-block distribution of spin characters. Section 3 presents some fundamental combinatorial concepts used in Sections 4 and 5. The theorem concerning the spin characters in a given 2-block is proved in Section 4, and in Section 5 we present our results on the decomposition numbers.

1. BACKGROUND AND PRELIMINARY RESULTS

For facts concerning the general representation theory of finite groups the reader is referred to [F, NT].

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Schur proved in 1911 that the finite symmetric groups S_n have covering groups $\hat{S}_n$ of order $2\,|S_n| = 2 \cdot n!$ [Sc]. This means that there is a non-split exact sequence

$$1 \to \langle z \rangle \to \hat{S}_n \xrightarrow{\pi} S_n \to 1$$

where $\langle z \rangle$ is a central subgroup of order 2 in $\hat{S}_n$.

Those irreducible characters of $\hat{S}_n$, which have $\langle z \rangle$ in their kernel, will be referred to as *ordinary characters*. The other irreducible characters of $\hat{S}_n$ are referred to as *spin characters*.

It is well-known that the ordinary characters of S_n are labeled canonically by the partitions $\lambda = (\ell_1, \ell_2, \cdots, \ell_m)$ of n; here $\ell_1 \geqslant \ell_2 \geqslant \cdots \geqslant \ell_m > 0$, $\ell_1 + \cdots + \ell_m = n$. The *length* $\ell(\lambda)$ of λ is defined as m. The set of partitions of n is denoted $\mathscr{P}(n)$ and for $\lambda \in \mathscr{P}(n)$, $[\lambda]$ denotes the corresponding ordinary character of $\hat{S}_n$ (resp., of S_n). For convenience we also write $\lambda \vdash n$ instead of $\lambda \in \mathscr{P}(n)$.

Let p be an arbitrary prime number. The distribution of the ordinary characters into p-blocks is described by a theorem, which is still frequently called the Nakayama Conjecture (see [JK, Theorem 6.1.21]). If $\lambda \in \mathscr{P}(n)$, let $\lambda_{(p)}$ denote its p-core, obtained from λ by removing successively all p-hooks from λ [JK, Theorem 2.7.16]. Then for $\lambda, \mu \in \mathscr{P}(n)$, $[\lambda]$ and $[\mu]$ are in the same p-block B of S_n if and only if $\lambda_{(p)} = \mu_{(p)}$. In this situation $|\lambda| - |\lambda_{(p)}|$ is a multiple of p, say $|\lambda| - |\lambda_{(p)}| = pw$. The integer w is an invariant of the block B, called the *weight* $w(B)$ of B.

Let $\mathrm{sgn} = [1^n]$ denote the sign character of S_n and $\hat{S}_n$. An irreducible character χ of $\hat{S}_n$ is called *self-associate* if $\chi \cdot \mathrm{sgn} = \chi$. Otherwise χ is called *non-self-associate* and χ and $\chi' = \chi \cdot \mathrm{sgn}$ are called a pair of associate characters.

For $\lambda \in \mathscr{P}(n)$, $[\lambda] \cdot \mathrm{sgn} = [\lambda^0]$ where λ^0 is the partition conjugate to λ [JK, 2.1.8].

The associate classes of spin characters of $\hat{S}_n$ are labeled canonically by the partitions of n into distinct parts, $\lambda = (\ell_1, \ell_2, \cdots, \ell_m)$, $\ell_1 > \ell_2 > \cdots > \ell_m > 0$, $\ell_1 + \ell_2 + \cdots + \ell_m = n$. We let $\mathscr{D}(n)$ denote the set of such partitions; for convenience we sometimes write $\lambda \succ n$ instead of $\lambda \in \mathscr{D}(n)$. Moreover,

$$\mathscr{D}^+(n) = \{\lambda = (\ell_1, \cdots, \ell_m) \in \mathscr{D}(n) \mid n - m \text{ even}\}$$

$$\mathscr{D}^-(n) = \{\lambda = (\ell_1, \cdots, \ell_m) \in \mathscr{D}(n) \mid n - m \text{ odd}\}.$$

The partitions in $\mathscr{D}^+(n)$ (resp., in $\mathscr{D}^-(n)$) are called *even* (resp. *odd*) partitions of n. To each $\lambda \in \mathscr{D}^+(n)$ corresponds a self-associate spin character $\langle \lambda \rangle$ and to each $\lambda \in \mathscr{D}^-(n)$ corresponds a pair $\langle \lambda \rangle$ and $\langle \lambda \rangle'$ of associate spin characters. We will make the convention that a statement/claim involving a non-self-associate character is made and proved simultaneously for its associate character, unless an exception is explicitly pointed out. For $\lambda \in \mathscr{D}(n)$ and p an odd prime, $\lambda_{(\bar{p})}$ denotes the $\bar{p}$-core (*p-bar-core*) of λ (see [MY, Ol2]. This means that $\lambda_{(\bar{p})}$ is obtained from λ by the inductive removal of p-bars, where the removal of a p-bar is a certain operation on the parts of λ. It was proved in [Hu, Ca] that the following holds:

If $\lambda = \lambda_{(\bar{p})}$, then $\langle \lambda \rangle$ is of p-defect 0.

If $\lambda, \mu \in \mathscr{D}(n)$, $\lambda \neq \lambda_{(\bar{p})}$, then $\langle \lambda \rangle$ and $\langle \mu \rangle$ are in the same p-block of $\hat{S}_n$ if and only if $\lambda_{(\bar{p})} = \mu_{(\bar{p})}$. (In particular, $\langle \lambda \rangle$ and $\langle \lambda \rangle'$ are in the same p-block.)

It should be mentioned that for p odd a p-block cannot contain ordinary and spin characters at the same time. In the case $p=2$, the characters in a 2-block B of S_n may be considered as the ordinary characters in a unique 2-block $\hat{B}$ of $\hat{S}_n$ (see [F, Chap. V, 4.5]). Then $\hat{B}$ also contains some spin characters. But neither the 2-core nor the $\bar{2}$-core (which may indeed be defined) of a $\lambda \in \mathscr{D}(n)$ tells us in which block $\langle \lambda \rangle$ is contained. In fact, the spin character $\langle 7 \rangle$ is not in the principal 2-block of $\hat{S}_7$. More generally, $\langle 4n+3 \rangle$ is not in the principal 2-block of $\hat{S}_{4n+3}$ [Ben]. A conjecture about how the spin characters should distribute into 2-blocks due to Knörr and the second author was stated in [Ol2]: Given $\lambda \in \mathscr{D}(n)$ define a partition $\mathrm{dbl}(\lambda) \in \mathscr{P}(n)$ by breaking each part ℓ_i of λ into two almost-equal parts. Thus if $\lambda = (8, 7, 4, 1)$ then $\mathrm{dbl}(\lambda) = (4, 4, 4, 3, 2, 2, 1)$. Then it was conjectured that the 2-core of $\mathrm{dbl}(\lambda)$ determines the block of $\langle \lambda \rangle$, i.e., $\langle \lambda \rangle$ and $[\mathrm{dbl}(\lambda)]$ should be in the same 2-block of $\hat{S}_n$. We prove this conjecture as our Theorem 4.1. Until now, only special cases of the conjecture had been verified. Theorem 1.2 in [Ben] implies that if λ is a "spin regular" partition, then $\langle \lambda \rangle$ is in the right block. This is but a consequence of the nonvanishing of a certain decomposition number. In Section 5 we generalize Benson's result to all $\lambda \in \mathscr{D}(n)$, using Theorem 4.1 and the methods of [BMO]. Although the statement of Theorem 4.1 resembles the Nakayama conjecture and the analogous statements for spin characters in odd characteristic or for unipotent characters of certain classes of finite linear groups in the non-defining characteristic [FS1, FS2], the methods applied in this paper to prove Theorem 4.1 are essentially different. Indeed, the principles of proof applied in the other cases cannot possibly work here. They involve in an essential way iterated versions of the (analogues of the) Murnaghan–Nakayama formula. These are used to evaluate character values on special p-singular elements, whose p-part has a large "weight." However, spin characters vanish on almost all 2-singular elements except those whose 2-part is z. The only other exception is one more 2-singular class where a given non-self-associate spin character does not vanish. This fact is indeed quite important for our investigation.

We need a few general facts from modular representation theory:

If G is a finite group and B a p-block of G, then $k(B)$ (resp. $\ell(B)$) denotes the number of ordinary irreducible (resp. modular irreducible) characters in B. When $u \in G$ is a p-element, then $Bl(C_G(u), B)$ is the set of p-blocks b of $C_G(u)$ s.t. $b^G = B$ (cf. [F, Chap. III, Section 9]). We have that $Bl(C_G(u), B) \neq \emptyset$, if and only if u is conjugate to an element in a defect group of B. We define

$$\ell_u(B) = \sum_{b \in Bl(C_G(u), B)} \ell(b).$$

The following is a well-known consequence of the Second Main Theorem on blocks:

$$k(B) = \sum_{u \in \mathscr{E}_p(G)} \ell_u(B), \tag{1.1}$$

where $\mathscr{E}_p(G)$ is a set of representatives for the conjugacy classes of p-elements in G.

Note that $\ell_1(B) = \ell(B)$.

A pair $s = (u, b_u)$, where $b_u \in Bl(C_G(u), B)$, is called a *subsection* for B. For each such subsection Brauer [Br] introduced a *matrix of contributions* $M^{(s)}$, a $k(B) \times k(B)$ complex matrix with rows and columns indexed by the irreducible characters in B (see also [F, Chap. V, Section 9]). If χ is an irreducible character in B, and s is as above, let

$$\chi^{(u, b_u)}(uv) = \chi^{(s)}(uv) = \sum_{\varphi} d^{(u)}_{\chi, \varphi} \varphi(v)$$

where the sum is taken over the modular irreducible characters φ in b_u. Here v is an element in $C_G(u)^0$, the set of p-regular elements in $C_G(u)$, and $d^{(u)}_{\chi, \varphi}$ is a generalized decomposition number. By the second main theorem on blocks

$$\chi(uv) = \sum_{b_u \in Bl(C_G(u), B)} \chi^{(u, b_u)}(uv) \tag{1.2}$$

in the above notation. In particular, if there is only one block in $Bl(C_G(u), B)$, then $\chi(uv) = \chi^{(u, b_u)}(uv)$.

When χ, ψ are irreducible characters in B and $s = (u, b_u)$ as above, then

$$m^{(s)}_{\chi, \psi} = \frac{1}{|C_G(u)|} \sum_{v \in C_G(u)^0} \chi^{(s)}(uv) \, \overline{\psi^{(s)}(uv)}$$

and

$$M^{(s)} = (m^{(s)}_{\chi\psi}).$$

Of all the interesting properties of this matrix ([Br, Sect. 5]) we need just one; in the above notation,

$$\mathrm{Tr}(M^{(s)}) = \ell(b_u). \tag{1.3}$$

2. THE NUMBER OF SPIN CHARACTERS IN THE 2-BLOCKS

For $n \in \mathbb{N}$ we need the following numbers:

$p(n) = \lvert \mathscr{P}(n) \rvert$	the number of partitions of n
$p^+(n)$, resp. $p^-(n)$	the number of even, resp. odd, partitions of n; i.e., the number of even, resp. odd, conjugacy classes of S_n
$q(n) = \lvert \mathscr{D}(n) \rvert$	the number of partitions of n into distinct parts
$q^+(n)$, resp. $q^-(n)$	the number of even, resp. odd, partitions of n into distinct parts

$$\tilde{p}^+(n) = \begin{cases} p^+(n) & n \text{ even} \\ p^-(n) & n \text{ odd} \end{cases}$$

$$\tilde{p}^-(n) = \begin{cases} p^-(n) & n \text{ even} \\ p^+(n) & n \text{ odd.} \end{cases}$$

Moreover, $\tilde{q}^+(n)$, $\tilde{q}^-(n)$ are defined analogously using $q^+(n)$, $q^-(n)$. Finally, $p_0(n)$ is the number of partitions of n into distinct odd parts and $\mathscr{P}_0(n)$ is the corresponding set of partitions.

We consider the two groups S_n and $\hat{S}_n$. For p-blocks B of S_n, the numbers $k(B)$, $\ell(B)$ depend only on the weight of the block. Indeed,

$$k(B) = k(p, w)$$

$$\ell(B) = k(p-1, w)$$

where for any $r, s \in \mathbb{N}$ we set

$$k(r, s) = \left| \left\{ (\lambda_1, ..., \lambda_r) \mid \lambda_1, ..., \lambda_r \text{ partitions, } \sum_{i=1}^{r} |\lambda_i| = s \right\} \right|.$$

Note. $k(1, s) = p(s) = |\mathscr{P}(s)|$.

The numbers $\ell_u(B)$ were computed in [Ol1] where it was also demonstrated how the knowledge of $k(B)$ for all B implies the knowledge of $\ell(B)$ for all B and vice versa. The same principle was used in [Ol3] for p-blocks $\hat{B}$ of spin characters in $\hat{S}_n$, for p odd, to compute $\ell(\hat{B})$, although no suitable labels are known for modular spin characters (except for $p = 3$ [BMO] and for $p=5$ [ABO]). In characteristic 2, if $B \subseteq \hat{B}$ where B is a 2-block of S_n and $\hat{B}$ a 2-block of $\hat{S}_n$, then $\ell(B) = \ell(\hat{B})$. Thus $\ell(\hat{B})$ is known for all $\hat{B}$ and this makes it possible to compute $k(\hat{B})$ for all $\hat{B}$.

In this section we prove:

THEOREM (2.1). *Let B be a 2-block of S_n of weight w and let $\hat{B}$ be the 2-block of $\hat{S}_n$ containing B. Then $\hat{B}$ contains $\tilde{p}^+(w)$ self-associate spin characters and $\tilde{p}^-(w)$ pairs of non-self-associate spin characters. Thus*

$$k(\hat{B}) - k(B) = p(w) + \tilde{p}^-(w).$$

THEOREM (2.2). *Notation is as in* (2.1). *Let B' be the block of A_n covered by B and let $\hat{B}'$ be the block of $\hat{A}_n$ covered by $\hat{B}$. Then*

$$k(\hat{B}') - k(B') = p(w) + \tilde{p}^+(w).$$

As a consequence of our considerations we prove at the end of this section a proposition of fundamental importance for Section 4.

Let $\pi: \hat{S}_n \to S_n$ be the canonical epimorphism with kernel $\langle z \rangle = Z(\hat{S}_n)$. When $x \in S_n$ let $\hat{x}$ denote one of the elements in $\hat{S}_n$ with $\pi(\hat{x}) = x$.

When $x \in S_n$ then the map $y \to [\hat{x}, \hat{y}]$ is a homomorphism from $C_{S_n}(x)$ to $\langle z \rangle$. Thus its kernel

$$C^*_{S_n}(x) = \{ y \in C_{S_n}(x) \mid [\hat{x}, \hat{y}] = 1 \}$$

is a subgroup of index at most 2 in $C_{S_n}(x)$. Indeed, by [Sc, Satz IV, p. 172] we have

LEMMA (2.3). *Let $x \in S_n$. Then*

$$|C_{S_n}(x) : C^*_{S_n}(x)| = \begin{cases} 1 & \text{if } x \text{ is odd and all cycles in } x \text{ have different length} \\ & \text{or } x \text{ is even and all cycles have odd length} \\ 2 & \text{otherwise} \end{cases}$$

Here, x is called odd resp. even if its cycle type is an odd resp. even partition of n.

Schur also showed [Sc, Satz III, p. 172]

LEMMA 2.4. *Let $x, y \in S_n$ be disjoint, i.e. their orbits of length >1 are disjoint. Then*

$$y \notin C^*(x) \Leftrightarrow \textit{Both } x \textit{ and } y \textit{ are odd.}$$

From now on, we consider a fixed 2*-element* u *of* S_n *and use the following notation*: For $i \geqslant 0$ we assume that u contains exactly m_i cycles of length 2^i, so that $\sum_{i \geqslant 0} 2^i m_i = n$.

The *weight of* u is then $w(u) = \sum_{i \geqslant 1} m_i 2^{i-1} = (n - m_0)/2$. It is convenient to consider $C(u) = C_{S_n}(u)$ as a direct product

$$C_{S_n}(u) = C^0(u) \times C^1(u),$$

where $C^0(u) \simeq S_{m_0}$ and $C^1(u) \cong \prod_{i \geqslant 1} \mathbb{Z}_{2^i} wr\, S_{m_i}$. The "base subgroup" E_u of $C(u)$ is the direct product of the base subgroups in the wreath products in $C^1(u)$, i.e.,

$$E_u = \prod_{i \geqslant 1} (\mathbb{Z}_{2^i})^{m_i}.$$

Thus E_u is simply the subgroup of $C(u)$ generated by the cycles of u.

LEMMA (2.5). (1) *If* u *is odd then* $E_u \subseteq C^*(u)$.

(2) *If* u *is even,* $u \neq 1$, *then* $E_u \nsubseteq C^*(u)$. *Indeed,* $E_u \cap C^*(u) = E'_u$ *consists of the even elements in* E_u.

Proof. Write $u = u_1 u_2$ where u_1 is one of the non-trivial cycles in u. If u is odd then u_2 is even and thus $u_1 \in C^*(u_2)$ by (2.4). It follows that $u_1 \in C^*(u)$. If u is even then u_2 is odd and by (2.4) $u_1 \notin C^*(u_2)$. We get $u_1 \notin C^*(u)$. Since the product of two elements in $C(u) \backslash C^*(u)$ is in $C^*(u)$ we get the last statement of (2).

LEMMA (2.6). *Suppose* $u = u_1 u_2 u'$ *where* u_1 *and* u_2 *are two nontrivial cycles of the same length, say* $u_1 = (1, 2, ..., 2^i)$, $u_2 = (2^i + 1, ..., 2 \cdot 2^i)$. *Let* $x = \prod_{j=1}^{2^i} (j, 2^i + j)$ *be the transposition in* $C(u)$ *interchanging* u_1 *and* u_2 *by conjugation. Then* $x \notin C^*(u)$.

Proof. Since x is even and disjoint to u' we have $x \in C^*(u')$ by (2.4). Thus it suffices to show $x \notin C^*(u_1 u_2)$. We consider x, u_1, and u_2 as elements of $S_{3 \cdot 2^i}$ and put $u_3 = (2 \cdot 2^i + 1, 2 \cdot 2^i + 2, ..., 3 \cdot 2^i)$. Thus, $v = u_1 u_2 u_3$ is odd and $x \in C(v)$. Consider $C(v) = \mathbb{Z}_{2^i} wr\, S_3$. By (2.3) $C^*(v)$ has index 2 in $C(v)$ and by (2.5) E_v, the base subgroup of $C(v)$, is contained in $C^*(v)$. This forces $C^*(v) = \mathbb{Z}_{2^i} wr\, A_3$. Since x corresponds to the element "(1, 2)" in S_3 we get $x \notin C^*(v)$. Since x and u_3 are disjoint and x is even, $x \in C^*(u_3)$. Thus $x \notin C^*(u_1 u_2)$, as desired.

LEMMA (2.7). *Let u be odd and assume that some $m_i \geqslant 2$. Then $E_u \subseteq C^*(u)$ and*

$$C(u)/E_u \simeq \prod_{i \geqslant 0} S_{m_i}.$$

Identifying $C(u)/E_u$ with $\prod_i S_{m_i}$ and considering $\prod_i S_{m_i}$ as a (Young) subgroup of $S_{\sum m_i}$, then $C^*(u)/E_u$ consists of the even elements in $\prod_i S_{m_i}$.

Proof. By (2.6) we see that the odd elements in an S_{m_i}, $i \geqslant 1$, are not in $C^*(u)$. The same is true for the odd elements in S_{m_0} (by (2.4)) since u is odd. The result follows.

LEMMA (2.8). *Let $u \neq 1$ be even. Then $E_u \nsubseteq C^*(u)$ and $E'_u = E_u \cap C^*(u) = E_u \cap C_{A_n}(u)$. We have*

$$C(u)/E_u \simeq C^*(u)/E'_u \simeq C_{A_n}(u)/E'_u \simeq \prod S_{m_i}.$$

Identifying $C_{A_n}(u)/E'$ with $\prod S_{m_i}$ and considering $\prod S_{m_i}$ as a (Young) subgroup of $S_{\sum m_i}$, then $C^*_{A_n}(u)/E'$ consists of the even elements in $\prod S_{m_i}$.

Proof. The statements about the isomorphisms are obvious, since $E_u C^*(u) = E_u C_{A_n}(u) = C_{S_n}(u)$. The odd elements of some S_{m_i}, $i \geqslant 1$, are not in $C^*_{A_n}(u)$ by (2.6) (although, considered as elements of S_n, they are even). The odd elements of S_{m_0} are obviously not in $C^*_{A_n}(u)$. The result follows.

LEMMA (2.9). *Let $u \neq 1$.*

(1) *All modular irreducible characters of $C_{S_n}(u)$ remain irreducible when restricted to $C_{A_n}(u)$.*

(2) *If u is even then all modular irreducible characters of $C_{S_n}(u)$ remain irreducible when restricted to $C^*(u)$.*

Proof. Since the nontrivial cycles of u are odd we have trivially that $E_u C_{A_n}(u) = C_{S_n}(u)$. By (2.8) $E_u C^*(u) = C_{S_n}(u)$, when u is even. We have

$$\ell(C(u)) = \ell(C(u)/E_u) = \ell(C_{A_n}(u)/E'_u) = \ell(C_{A_n}(u))$$

since E_u, E'_u are normal 2-subgroups of $C(u)$, $C_{A_n}(u)$. Similar it is proved that $\ell(C(u)) = \ell(C^*(u))$ when u is even.

Let b_u be a 2-block of $C_{S_n}(u)$. We write $b_u = b_u^{(0)} \times b_u^{(1)}$ where $b_u^{(0)}$ is a block of $C^0(u) = S_{m_0}$ and $b_u^{(1)}$ is the unique block of $C^1(u)$ ($C^1(u)$ has a unique block, since E_u is a self-centralizing normal 2-subgroup in $C^1(u)$). We denote by w_0 the weight of the block $b_u^{(0)}$.

Let b_u^* be a block of $C^*(u)$ covered by b_u. By (2.7) and (2.9) b_u^* is unique, when $u \neq 1$ and $w_0 \neq 0$. When $u \neq 1$ is even we consider also the block $\tilde{b}_u$ of $C_{A_n}(u)$ covered by b_u (unique by (2.9)) and the block $\tilde{b}_u^*$ of $C_{A_n}^*(u)$ covered by b_u^*. This block is unique by (2.8) except when $w_0 = 0$ and $m_i \leqslant 1$ for all $i \geqslant 1$, and it is also the unique block covered by $\tilde{b}_u$.

LEMMA (2.10). *Let $u \neq 1$.*

(1) *In the following cases $\ell(b_u^*) = \ell(b_u)$:*

- (i) *u even.*
- (ii) *u odd and w_0 odd.*
- (iii) *u odd and all $m_i \leqslant 1$.*
- (iv) *u odd, $w_0 = 0$ and $m_i \leqslant 1$ for $i \geqslant 1$.*

(2) *Otherwise*

$$\ell(b_u^*) - \ell(b_u) = p\left(\frac{w_0}{2}\right) \prod_{i \geqslant 1} p_0(m_i).$$

Proof. If u is even, the claim follows from Lemma 2.9 (2). We assume u is odd. If all $m_i \leqslant 1$ then $C^*(u) = C(u)$ so that $b_u^* = b_u$. If $w_0 = 0$ and $m_i \leqslant 1$ for $i \geqslant 1$ then b_u contains a unique modular character ψ. The two constituents of $\psi_{|C^*(u)}$ are in b_u^* and in another block of $C^*(u)$. Thus $\ell(b_u^*) = 1$. (This is the only case where b_u covers two blocks of $C^*(u)$.) Otherwise E_u is a normal 2-subgroup of $C^*(u)$ and of $C(u)$ and the modular characters of b_u may be considered as modular characters $\psi = \prod \psi_i$ of $X = \prod_{i \geqslant 0} S_{m_i} \simeq C(u)/E_u$ for which $\psi_0 \in b_u^{(0)}$ and ψ_i, $i \geqslant 1$, is arbitrary. By (2.7) $Y = C^*(u)/E_u$ may be considered as the subgroup of even elements in X. Therefore the modular character ψ splits when restricted to Y if and only if all ψ_i, $i \geqslant 1$, split when restricted to A_{m_i} (when $m_i \geqslant 2$). Obviously $\ell(b_u^*) - \ell(b_u)$ equals the number of characters which split. When w_0 is odd, no modular character of $b_u^{(0)}$ splits [Ol2, (2.17) (1)]. Thus again $\ell(b_u^*) = \ell(b_u)$. When w_0 is even, then exactly $p(w_0/2)$ modular characters of $b_u^{(0)}$ split [Ol3, (2.17) (2)]. For $i \geqslant 1$ and $m_i \geqslant 2$ the number of ψ_i which split is obviously $\ell(A_{m_i}) - \ell(S_{m_i})$, which, as is well-known, equals $p_0(m_i)$.

LEMMA (2.11). *Let $u \neq 1$ be even.*

(1) *In the following cases $\ell(\tilde{b}_u^*) = \ell(b_u^*)$:*

- (i) *w_0 is odd.*
- (ii) *All $m_i \leqslant 1$.*
- (iii) *$w_0 = 0$ and $m_i \leqslant 1$ for $i \geqslant 1$.*

(2) *Otherwise,*

$$\ell(\tilde{b}_u^*) - \ell(b_u^*) = p\left(\frac{w_0}{2}\right) \prod_{i \geqslant 1} p_0(m_i).$$

Proof. This is analogous to the proof of (2.10) using (2.8) in place of (2.7).

We now let $\mathscr{E}_2(n)$ be a set of representatives for the conjugacy classes of 2-elements in S_n, and we let $\mathscr{E}_2(\hat{n})$ be a similar set for $\hat{S}_n$. We assume they are chosen such that for all $v \in \mathscr{E}_2(\hat{n})$, $\pi(v) \in \mathscr{E}_2(n)$.

Let B be a 2-block of S_n and let $\hat{B}$ be the 2-block of $\hat{S}_n$ containing B (cf. [F, Chap. V, 4.5]). For $u \in \mathscr{E}_2(n)$ we define

$$s_u(\hat{B}) = \left(\sum_{\{v \in \mathscr{E}_2(n) \mid \pi(v) = u\}} \ell_v(\hat{B}) \right) - \ell_u(B).$$

(For any u, B: $\ell_u(B) = \sum_{b \in Bl(C(u), B)} \ell(b)$). By Brauer's formula (1.1) we get that

$$k(\hat{B}) - k(B) = \sum_{u \in \mathscr{E}_2(n)} s_u(\hat{B}). \tag{2.12}$$

We note the following facts:

(I) Suppose that $v \in \mathscr{E}_2(\hat{n})$ and that $\pi(v) = u \neq 1$. Suppose that b_v is a block of $C_{\hat{S}_n}(v)$. The block b_v contains a block b_u^* of $C_{\hat{S}_n}(v)/\langle z \rangle = C^*(u)$. If b_u is a block of $C(u)$ covering b_u^* then

$$b_v^{\hat{S}_n} = \hat{B} \qquad \text{if and only if} \quad b_u^{S_n} = B.$$

This is an easy consequence of the definition of "induced" blocks.

(II) There exists a block b_u of $C_{S_n}(u)$ with $b_u^{S_n} = B$ if and only if $w(u) \leqslant w(B) = w$ (see [Pu] or [Ol1]). Then $w_0 = w(b_u^{(0)}) = w(B) - w(u)$.

(III) $s_1(\hat{B}) = \ell(B)$.

Indeed, $\{v \in \mathscr{E}_2(\hat{n}) \mid \pi(v) = 1\} = \{1, z\}$. But $\ell_1(\hat{B}) = \ell_z(\hat{B}) = \ell_1(B) = \ell(B) = p(w)$.

Lemma (2.13). *Let $u \neq 1$.*

(1) *If u is even or if $w(B) - w(u)$ is odd, then $s_u(\hat{B}) = 0$.*

(2) *Otherwise*

$$s_u(\hat{B}) = p\left(\frac{w(B) - w(u)}{2}\right) \prod_{i \geqslant 1} p_0(m_i).$$

Proof. If there exists an i, s.t. $m_i \geqslant 2$, the statement follows from (2.10) except in the case where u is odd, $w_0 = 0$, and $m_i \leqslant 1$ for $i \geqslant 1$. Then b_u covers two blocks of $C^*(u)$. By the above there are two blocks in $Bl(C_{\hat{S}_n}(v), \hat{B})$, each with exactly one modular character. We get $s_u(\hat{B}) = 1$. If u is odd and all $m_i \leqslant 1$ then $C^*(u) = C(u)$ and $b_u^* = b_u$. But in this case, if $\pi(v) = u$, then v and vz are not conjugate in $\hat{S}_n$ and again we get $s_u(\hat{B}) = 1$.

Continuing our study of B and $\hat{B}$, we consider also the blocks B' and $\hat{B}'$ where B' is a block of A_n covered by B and $\hat{B}'$ is the block of $\hat{A}_n$ containing B'. Thus $\hat{B}$ covers $\hat{B}'$. Except in the case $w(B) = 0$ which is trivial for us, B' and $\hat{B}'$ are unique.

For $u \in \mathscr{E}_2(n)$ *even* we define

$$t_u(\hat{B}') = \left(\sum_{\{v \in \mathscr{E}_2(n) \mid \pi(v) = u\}} \ell_v(\hat{B}') \right) - \ell_u(B').$$

By Brauer's formula (1.1) we get that

$$k(\hat{B}') - k(B') = \sum_{\substack{u \in \mathscr{E}_2(n) \\ u \text{ even}}} t_u(\hat{B}'). \tag{2.14}$$

We note:

(IV) Let $v \in \mathscr{E}_2(u)$, s.t. $u = \pi(v) \neq 1$ is even. Assume that $\tilde{b}_v$ is a block of $C_{\hat{A}_n}(v)$. This block contains a block $\tilde{b}_u^*$ of $C_{\hat{S}_n}(v)/\langle z \rangle = C_{A_n}^*(u)$. If $\tilde{b}_u$ is a block of $C_{A_n}(u)$ covering $\tilde{b}_u^*$ then

$$\tilde{b}_v^{\hat{A}_n} = \hat{B}' \Leftrightarrow \tilde{b}_u^{A_n} = B'.$$

(V) If u is even there exists a block $\tilde{b}_u$ of $C_{A_n}(u)$ with $\tilde{b}_u^{A_n} = B'$ if and only if $w(u) \leqslant w(B)$.

(VI)

$$t_1(\hat{B}') = \ell(B') = \begin{cases} p(w) & w \text{ odd} \\ p(w) + p(w/2) & w \text{ even} \end{cases}$$

This follows from [Ol3, (2.17)].

LEMMA (2.15). *Let $u \neq 1$ be even.*

(1) *If $w(B) - w(u)$ is odd then $t_u(\hat{B}') = 0$.*

(2) *Otherwise*

$$t_u(\hat{B}') = p\left(\frac{w(B) - w(u)}{2}\right) \prod_{i \geqslant 1} p_0(m_i).$$

Proof. Analogous to the proof of (2.13) using (2.11) in place of (2.10).

To be able to sum $s_u(\hat{B})$ and $t_u(\hat{B}')$ over all relevant u we need two combinatorial lemmas.

LEMMA (2.16). *Let $w_1 \geqslant 0$. Then*

$$(1) \qquad \sum_{\substack{(m_1, m_2, \ldots),\, m_i \geqslant 0 \\ \sum m_i 2^{i-1} = w_1,\, \sum m_i \text{ odd}}} \prod_{i \geqslant 1} p_0(m_i) = \tilde{q}^-(w_1)$$

$$(2) \qquad \sum_{\substack{(m_1, m_2, \ldots),\, m_i \geqslant 0 \\ \sum m_i 2^{i-1} = w_1,\, \sum m_i \text{ even}}} \prod_{i \geqslant 1} p_0(m_i) = \tilde{q}^+(w_1)$$

Proof. We exhibit bijections to prove this result. Let $\lambda^{(1)}, \lambda^{(2)}, \ldots$ be a sequence of partitions with $\lambda^{(i)} \in \mathscr{P}_0(m_i)$. Then

$$\lambda = \bigcup_{i \geqslant 1} 2^{i-1} \cdot \lambda^{(i)}$$

is a partition of $w_1 = \sum 2^{i-1} m_i$. Thus $2^j x_j$, x_j odd, is a part of λ if and only if x_j is a part of $\lambda^{(j+1)}$. The map

$$(\lambda^{(i)}) \mapsto \lambda$$

is a bijection

$$\bigcup_{\substack{(m_i) \\ \sum m_i 2^{i-1} = w_1}} \prod_i \mathscr{P}_0(m_i) \rightarrow \mathscr{D}(w_1).$$

Since each $\lambda^{(i)}$ has only odd parts we have for $\lambda^{(i)} \in \mathscr{P}_0(m_i)$ that

$$m_i \equiv \ell(\lambda^{(i)}) \qquad \text{mod } 2.$$

If $(\lambda^{(i)}) \mapsto \lambda$ we have

$$\ell(\lambda) = \sum_i \ell(\lambda^{(i)}) \equiv \sum_i m_i \qquad (\text{mod } 2).$$

From this (1) and (2) follow.

LEMMA (2.17). *Let ε be a sign. Then $\tilde{p}^\varepsilon(w) = \sum_{w' \geqslant 0} p(w')\, \tilde{q}^\varepsilon(w - 2w')$.*

Proof. This may be proved using generating functions and [Ol3, (2.9)]. We give here a bijective proof.

Given a partition λ of w' and a partition μ of $w-2w'$ in distinct parts, we define a partition ρ of w as follows:

$$\rho = (\lambda)^2 \cup \mu = \text{the partition with } \lambda\text{-parts with double multiplicity and with } \mu\text{-parts.}$$

Obviously, ρ is a partition of w. Since the parts of λ are taken twice for ρ, clearly ρ and μ have the same sign.

Given ρ we may recover λ and μ as follows. If $\rho = (i^{m_i})$, write $m_i = \varepsilon_i + 2m_i'$, $\varepsilon_i \in \{0, 1\}$, then $\lambda = (i^{m_i'})$, $\mu = (i^{\varepsilon_i})$.

LEMMA (2.18). *We have*

$$(1) \quad \sum_{u \neq 1} s_u(\hat{B}) = \tilde{p}^-(w)$$

$$(2) \quad \sum_{\substack{u \neq 1 \\ \text{even}}} t_u(\hat{B}') = \begin{cases} \tilde{p}^+(w) & w \text{ odd} \\ p^+(w) - p(w/2) & w \text{ even.} \end{cases}$$

Proof. (1) $s_u(\hat{B}) \neq 0$ if and only if $w_0 = w(B) - w(u)$ is even and u is *odd*. Moreover, if u contains m_i cycles of length 2^i then u is odd if and only if $\sum_{i \geqslant 1} m_i$ is odd. Given $w' \geqslant 0$ we collect all *odd* u with $w(B) - w(u) = 2w'$. Then by (2.13), (2.16)(1), and (2.17) with $\varepsilon = -1$ we get ($w = w(B)$)

$$\sum_{u \neq 1} s_u(\hat{B}) = \sum_{w' \geqslant 0} p(w')\, \tilde{q}^-(w - 2w') = \tilde{p}^-(w).$$

(2) is proved similarly using (2.15), (2.16)(2), and (2.17) with $\varepsilon = +1$. Since $u = 1$ is even, the exclusion of $u = 1$ in the summation means that when w is even, then the summand for $w' = w/2$ is missing in (2.17). The exclusion of $u = 1$ does not create a problem, when w is odd, since $\tilde{q}^+(1) = 0$. Therefore the summand in (2.17) for $w' = (w-1)/2$ is 0.

Proof of (2.1) *and* (2.2). Let $w = w(B)$. As we have seen in (III) and (VI),

$$s_1(\hat{B}) = p(w)$$

$$t_1(\hat{B}) = \begin{cases} p(w) & w \text{ odd} \\ p(w) + p(w/2) & w \text{ even.} \end{cases}$$

By (2.12) and (2.18)(1) we then get

$$k(\hat{B}) - k(B) = \sum_u s_u(\hat{B}) = p(w) + \tilde{p}^-(w) = \tilde{p}^+(w) + 2\tilde{p}^-(w).$$

Similarly, by (2.14) and (2.18)(2) we get

$$k(\hat{B}') - k(B') = p(w) + \tilde{p}^{+}(w) = \tilde{p}^{-}(w) + 2\tilde{p}^{+}(w).$$

Thus in both blocks $\hat{B}$ and $\hat{B}'$ the number of spin characters depend in an explicit way only on $w = w(B)$. Therefore the statements about the number of s.a. (n.s.a.) characters follow.

We finish this section by proving a proposition which will be an essential ingredient in the inductive proof of our main Theorem (4.1).

PROPOSITION (2.19). *Let $\lambda \in \mathscr{D}^{-}(n)$ and suppose that $\langle\lambda\rangle \in \hat{B}$, where $\hat{B}$ contains the 2-block B of S_n. Then, if $w^*(\lambda)$ is the 2-singular weight of λ (i.e., $w^*(\lambda) = \frac{1}{2}\sum_{\ell_i \text{ even part of } \lambda} \ell_i$) then $w^*(\lambda) \equiv w(B)$* (mod 2).

Proof. Let x be an element of S_n of cycle type λ, and let u be its 2-singular part. This means that $w(u) = w^*(\lambda)$. Since $\lambda \in \mathscr{D}^{-}(n)$, u is odd. By [Sc, p. 236], $\langle\lambda\rangle(\hat{x}) \neq 0$. Therefore $\hat{u}$ is contained in a defect group of $\hat{B}$, and thus u is contained in a defect group of B. This forces $w(u) \leqslant w(B)$. Let $s = (u, b_u)$ be the unique u-subsection for B. As before let $b_u = b_u^{(0)} \times b_u^{(1)}$ so that $w(B) - w(u) = w(b_u^{(0)}) = w_0$ [Pu, Ol1]. Unless $w_0 = 0$ and $m_i \leqslant 1$ for all $i \geqslant 1$, b_u covers a unique block b_u^* of $C^*(u)$. But in the case where $w_0 = 0$ there is nothing to prove. Thus we may assume that if $b_{\hat{u}}$ is the block of $C_{\hat{S}_n}(\hat{u})$ containing b_u^*, then $\hat{s} = (\hat{u}, b_{\hat{u}})$ is the unique $\hat{u}$-subsection of $\hat{B}$.
This forces (by (1.2))

$$\langle\lambda\rangle^{(\hat{s})}(\hat{x}) = \langle\lambda\rangle(\hat{x}).$$

Thus $\langle\lambda\rangle^{(\hat{s})}(\hat{x}) \neq 0$, which means that the diagonal element $m^{(\hat{s})}_{\langle\lambda\rangle\langle\lambda\rangle}$ in the matrix $M^{(\hat{s})}$ of contributions is non-zero. In fact, it equals $\frac{1}{2}$(!). Since $B \subseteq \hat{B}$, the matrix $M^{(s)}$ of contributions is the submatrix of $M^{(\hat{s})}$ containing only contributions of the ordinary characters in $\hat{B}$. By (1.3),

$$\operatorname{Tr} M^{(s)} = \ell(b_u), \qquad \operatorname{Tr} M^{(\hat{s})} = \ell(b_{\hat{u}}) = \ell(b_u^*).$$

Since the diagonal element $m^{(\hat{s})}_{\langle\lambda\rangle\langle\lambda\rangle} \neq 0$ we get that $\operatorname{Tr} M^{(s)} \neq \operatorname{Tr} M^{(\hat{s})}$, i.e., $\ell(b_u) \neq \ell(b_u^*)$. By (2.10) this means that w_0 is even, as claimed.

Remark. It will be a consequence of our main result that the statement of (2.19) is in fact also true if $\lambda \in \mathscr{D}^{+}(n)$, i.e., when $\langle\lambda\rangle$ is self-associate. In this case $\langle\lambda\rangle$ is in fact 0 on all 2-singular elements $\neq z$, so that the argument in the proof of (2.19) cannot be applied. If (2.19) could in some other way be proved directly also for $\lambda \in \mathscr{D}^{+}(n)$, our inductive proof of the main theorem would become much shorter.

3. COMBINATORIAL CONCEPTS

First we briefly recall a few facts from the representation theory of S_n and the relevant combinatorial concepts (see [JK]).

Instead of computing the 2-core of a partition $\alpha \vdash n$ (as described in [JK]) there is an alternative way for determining the 2-block of S_n to which the character $[\alpha]$ belongs. For this, fill the (i, j)-nodes of the Young diagram of α with their 2-residues $j-i \pmod 2$; for example,

$$\alpha=(4, 3, 1^2): \quad \begin{array}{llll} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & \\ 0 & & & \\ 1 & & & \end{array}$$

We call this the 2-*residue diagram of* α and define

$$\begin{aligned} \delta(\alpha) := {} & \#\{0\text{'s in the 2-residue diagram of } \alpha\} \\ & - \#\{1\text{'s in the 2-residue diagram of } \alpha\}. \end{aligned}$$

Removing 2-hooks from the Young diagram of α amounts to removing a (0 1)-pair from its 2-residue diagram, so that the difference of the numbers of 0's and 1's is unchanged in the process and

$$\delta(\alpha)=\delta(\alpha_{(2)}),$$

where $\alpha_{(2)}$ denotes the 2-core of α.

Now we consider all possible 2-cores to show that they are distinguished by the δ-numbers. The non-empty 2-cores are of the form $\kappa_k=(k, k-1, \cdots, 2, 1)$; one easily checks that

$$\delta(\emptyset)=0, \qquad \delta(k, k-1, \cdots, 1)=(-1)^{k+1}\left[\frac{k+1}{2}\right].$$

Hence the 2-block of $[\alpha]$ is determined by $\delta(\alpha)$, which is equivalent to knowing the *content* $c(\alpha)$ of α, where

$$\begin{aligned} c(\alpha)=(& \#\{0\text{'s in the 2-residue diagram of } \alpha\}, \\ & \#\{1\text{'s in the 2-residue diagram of } \alpha\}). \end{aligned}$$

By the Branching Theorem [JK, 2.4.3], the constituents of $[\alpha]\uparrow^{S_{n+1}}$ are obtained by adding a node to the diagram of α. By the above, the constituents fall into (at most) two 2-blocks depending on the 2-residue of the added node. We will describe explicitly which 2-blocks of S_{n+1} lie over a given 2-block of S_n, and respectively which 2-blocks of S_{n-1} lie under it.

PROPOSITION (3.1). *Denote by B_n^k the unique 2-block of S_n with 2-core κ_k if $k>0$, respectively with 2-core $\varnothing$ if $k=0$ (if such a block exists). Then*

$$\text{(a)}\quad B_n^k\uparrow^{S_{n+1}} \textit{ is contained in } \begin{cases} B_{n+1}^{k+2}\cup B_{n+1}^{k-2} & \textit{if}\quad k\geqslant 2\\ B_{n+1}^{0}\cup B_{n+1}^{3} & \textit{if}\quad k=1\\ B_{n+1}^{1}\cup B_{n+1}^{2} & \textit{if}\quad k=0 \end{cases}$$

$$\text{(b)}\quad B_n^k\downarrow_{S_{n-1}} \textit{ is contained in } \begin{cases} B_{n-1}^{k+2}\cup B_{n-1}^{k-2} & \textit{if}\quad k\geqslant 2\\ B_{n-1}^{0}\cup B_{n-1}^{3} & \textit{if}\quad k=1\\ B_{n-1}^{1}\cup B_{n-1}^{2} & \textit{if}\quad k=0 \end{cases}$$

Proof (a) As noted above, the δ-invariant for B_n^k is (interpreting κ_k as $\varnothing$ for $k=0$)

$$\delta(\kappa_k)=(-1)^{k+1}\left[\frac{k+1}{2}\right].$$

Adding a 0- resp. 1-node implies that the δ-invariants of the 2-blocks of S_{n+1} "over" B_n^k are

$$(-1)^{\varepsilon}+(-1)^{k+1}\left[\frac{k+1}{2}\right],\qquad \varepsilon\in\{0,1\}.$$

For $k\geqslant 2$, these two numbers equal

$$\delta(\kappa_{k+2})=(-1)^{k+3}\left[\frac{k+3}{2}\right],\qquad \delta(\kappa_{k-2})=(-1)^{k-1}\left[\frac{k-1}{2}\right].$$

For $k=1$, the δ-invariants above are $\pm 1+1$, i.e., 0 and 2, which are $\delta(\varnothing)$ and $\delta(\kappa_3)$.

For $k=0$, the δ-invariants above are ± 1, i.e., $\delta(\kappa_1)$ and $\delta(\kappa_2)$. Hence (a) is proved, and (b) follows in the same way (or by Frobenius reciprocity).

Remark 3.2. We recall that the weight of the block B_n^k is $w = (n - \binom{k+1}{2})/2$. Then B_{n+1}^{k+2} and B_{n+1}^{k-2} are of weight $w-(k+1)$ and $w-k$, respectively, for $k \geqslant 2$; B_{n+1}^0 and B_{n+1}^3 are of weight $w+1$ and $w-2$, respectively, for $k=1$; and B_{n+1}^1 and B_{n+1}^2 are of weight w and $w-1$, respectively, for $k=0$.

Similarly, B_{n-1}^{k+2} and B_{n-1}^{k-2} are of weight $w-(k+2)$ and $w+(k-1)$, respectively, for $k \geqslant 2$; B_{n-1}^0 and B_{n-1}^3 are of weight w and $w-3$, respectively, for $k=1$; and B_{n-1}^1 and B_{n-1}^2 are of weight $w-1$ and $w-2$, respectively, for $k=0$. The main observation to make is that in all cases the two blocks lying over B_n^k, respectively "under" B_n^k, have non-congruent weights modulo 2.

Furthermore, note that the corresponding 2-blocks $\hat{B}_n^k \supseteq B_n^k$ on the $\hat{S}_n$-level satisfy the very same relations with respect to induction and restriction, as the 2-modular representations of S_n and $\hat{S}_n$ are the same.

For the later purpose of investigating the decomposition matrix we also have to recall the concepts of ladders and regularization of partitions. We consider again the 2-residue diagram:

$$
\begin{array}{ccccccccccccccccc}
0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 \\
& / & & / & & / & & / & & / & & / & & / & & / & \\
1 & & 0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & \\
& / & & / & & / & & / & & / & & / & & / & & & \\
0 & & 1 & & 0 & & 1 & & 0 & & 1 & & 0 & & & & \\
& / & & / & & / & & / & & / & & & & & & & \\
1 & & 0 & & 1 & & 0 & & 1 & & & & & & & & \\
& / & & / & & / & & & & & & & & & & & \\
0 & & 1 & & 0 & & & & & & & & & & & & \\
& / & & & & & & & & & & & & & & & \\
1 & & & & & & & & & & & & & & & &
\end{array}
$$

The *ladders* in this diagram are indicated by the lines joining the 0's and 1's. More precisely, the 0-ladders connect the nodes (from bottom to top):

$$L_{i0}\colon (2i-1, 1) \to (2i-2, 2) \to (2i-3, 3) \to \cdots \to (1, 2i-1)$$

and the 1-ladders connect the nodes (from bottom to top):

$$L_{i1}\colon (2i, 1) \to (2i-1, 2) \to (2i-2, 3) \to \cdots \to (1, 2i).$$

The ladders in a partition α are just the intersections of the L_{ij} with (the 2-residue diagram of) α. It is clear that the 2-regular partitions are exactly those partitions α where all the nodes on the ladders of α form top parts of these ladders. Given an arbitrary partition α, we "regularize" α by replacing the nodes in each ladder $L_{ij}(\alpha)$ by the same number of nodes at the top of L_{ij}; it is easy to check that this gives again a partition which we call α^R (see [JK, p. 282]).

EXAMPLE. $\alpha = (4^2, 3, 1^2)$

$$\begin{array}{llll} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \\ 1 & & & \\ 0 & & & \end{array} \quad \rightarrow \quad \begin{array}{lllll} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & \\ 0 & 1 & 0 & & \\ 1 & & & & \\ & & & & \end{array} \quad , \text{ so } \alpha^R = (5, 4, 3, 1).$$

We now have to develop suitable analogues for dealing with spin characters at characteristic 2. In odd characteristic p the right combinatorial concepts are the $\bar{p}$-abacus and the shifted $\bar{p}$-residue diagram (see [Mo]); unfortunately, these tools cannot be extended to the case $p = 2$. We thus have to introduce some new notions.

First we define a suitable abacus for computing the appropriate core of a bar partition, called the $\bar{4}$-*abacus*. This abacus has 3 runners, labeled 0/2, 1, 3 from left to right. The runners 1 and 3 are called *conjugate runners*. The numbers 0, 1, 2, ... are then placed in the abacus as follows:

0/2	1	3
0	1	3
2	5	7
4	9	11
6	13	15
8	⋮	⋮
⋮	⋮	⋮ .

The bead configuration of a bar partition $\lambda = (\ell_1, \cdots, \ell_m) \in \mathscr{D}(n)$ is obtained by circling the integers (in position) $\ell_1, \cdots, \ell_m$.

EXAMPLE. $\lambda=(13, 11, 8, 5, 2)\in\mathscr{D}(39)$ has the configuration:

$$\begin{array}{ccc} 0 & 1 & 3 \\ 2 & 5 & 7 \\ 4 & 9 & 11 \\ 6 & 13 & \\ 8 & & \end{array} .$$

Starting with such a configuration, we can now operate on the abacus using the following two types of moves:

(i) sliding a bead one position up the runner (if the new position is not yet occupied); beads in position 0 are dumped.

(ii) removing the beads on conjugate runners in the top row.

Operating according to these rules as long as possible we finally obtain a configuration called the $\bar{4}$-core of λ, denoted $\lambda_{(\bar{4})}$. If $\lambda_{(\bar{4})}$ is a partition of r, then $n=r+2w$ and w is called the *weight* (the $\bar{4}$-weight) of λ. In the example above, the final configuration is

$$\begin{array}{ccc} 0 & 1 & 3 \\ 2 & 5 & 7 \\ \vdots & \vdots & \vdots \end{array} ,$$

i.e., the $\bar{4}$-core of $(13, 11, 8, 5, 2)$ is just (1) and the weight of λ is 19.

It is clear that the non-empty $\bar{4}$-cores are of the form

$$(4k+1, 4k-3, \cdots, 5, 1) \quad \text{resp.} \quad (4k+3, 4k-1, \cdots, 7, 3), \qquad k\in\mathbb{N}_0.$$

PROPOSITION (3.3). *Let $\rho\in\mathscr{D}(r)$ be a fixed $\bar{4}$-core, $w\in\mathbb{N}_0$, ε a sign, and $n=r+2w$. Set*

$$k^{\varepsilon}(\rho; n)=\#\{\lambda\in\mathscr{D}^{\varepsilon}(n)\mid\lambda_{(\bar{4})}=\rho\}.$$

Then

$$\tilde{p}^{\varepsilon}(w)=k^{\varepsilon}(\rho; n).$$

Proof. We construct a bijection

$$\mathscr{P}(w)\to\{\lambda\in\mathscr{D}(n)\mid\lambda_{(\bar{4})}=\rho\}=\mathscr{D}_{\rho}(n)$$

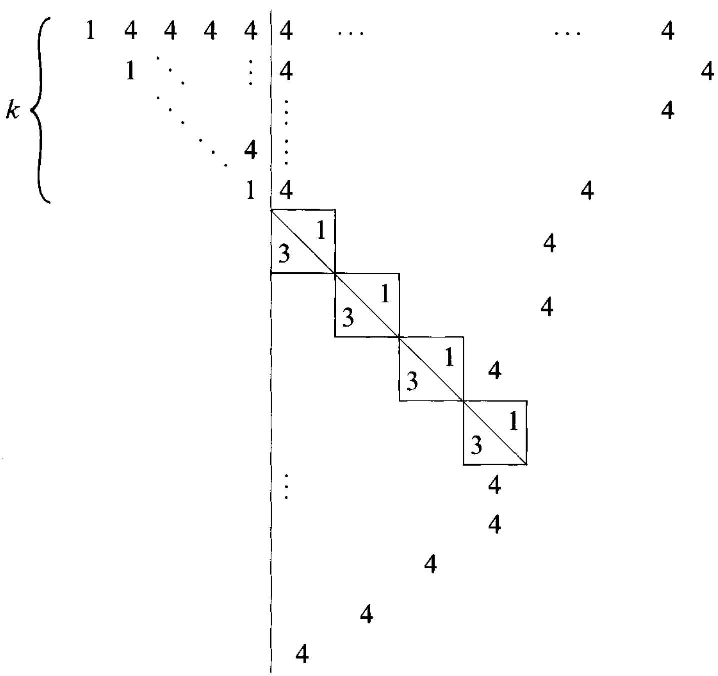
and check that the "signs" are right.

Take $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, ..., \ell^{\alpha_\ell}) \in \mathcal{P}(w)$, written exponentially, say $\alpha_i = \alpha_i' + 2\alpha_i''$ with $\alpha_i' \in \{0, 1\}$. Set $\alpha' = (1^{\alpha_1'}, 2^{\alpha_2'}, ..., \ell^{\alpha_\ell'})$ (a partition into distinct parts) and $\alpha'' = (1^{\alpha_1''}, 2^{\alpha_2''}, ..., \ell^{\alpha_\ell''})$.

We now construct an associated partition $\lambda \in \mathcal{D}_\rho(n)$ for which α' is the quotient of the 0/2-runner, giving the even parts of λ, and α'' is the bar-quotient for the 1- and 3-runner, giving the odd parts of λ. The associated partition λ has the following parts:

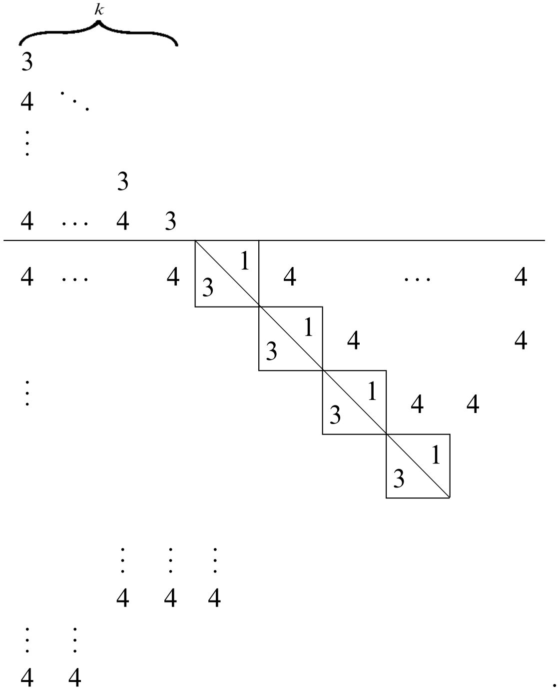
(i) $2^{\alpha_1'}, 4^{\alpha_2'}, \cdots, (2\ell)^{\alpha_\ell'}$.

(ii) If $\rho = (4k-1, \cdots, 5, 1)$ or $\varnothing$, consider the following diagram built from α'' and ρ, where the partition to the right (of the line) has shape α'', with the boxes filled by 4's.



Then the parts $\equiv 1 \pmod 4$ of λ are obtained by reading the rows starting at 1 (as long as there is a 1 in the row), and the parts $\equiv 3 \pmod 4$ of λ are obtained by reading the columns starting at 3 (if there is a 3 in the column).

Analogously, if $\rho = (4+3, ..., 7, 3)$, we put ρ "on top" of α'' and use the same reading rule:



By construction, $\lambda_{(\bar{4})} = \rho$ and λ is a bar partition of

$$r + 2\,|\alpha'| + 4\,|\alpha''| = r + 2\,|\alpha| = r + 2w = n.$$

Given $\lambda \in \mathscr{D}_\rho(n)$, it is obvious from the above how to construct the associated partition $\alpha \in \mathscr{P}(w)$.

We only have to show that the signs are correct:

$$\begin{aligned}
\#\{\text{even parts of } \lambda\} &= \sum_{i \geqslant 1} \alpha'_i \equiv \sum_{i \geqslant 1} \alpha_i \qquad (\text{mod } 2) \\
&\equiv \sum_{i \geqslant 1} \alpha_{2i} + \sum_{i \geqslant 1} i\alpha_i \qquad (\text{mod } 2) \\
&= \sum_{i \geqslant 1} \alpha_{2i} + w \\
&= \#\{\text{even parts of } \alpha\} + w.
\end{aligned}$$

Defining the sign $\varepsilon(\alpha)$ of a partition α by

$$\varepsilon(\alpha) = \begin{cases} 0 & \text{if} \quad \alpha \text{ is even} \\ 1 & \text{if} \quad \alpha \text{ is odd,} \end{cases}$$

we thus have $\varepsilon(\alpha) + w \equiv \varepsilon(\lambda)$ (mod 2), proving the claim.

In the rest of this section we want to study the relationship between the "$\bar{4}$-combinatorics" of bar partitions and the "2-combinatorics" of arbitrary partitions. First we have to define a doubling process for spin partitions:

DEFINITION (3.4). Given a bar partition $\lambda = (\lambda_1 > \cdots > \lambda_m) \in \mathscr{D}(n)$, we replace each part λ_i of λ by the parts $[(\lambda_i + 1)/2]$, $[\lambda_i/2]$, where $[\ ]$ denotes "integral part of". Thus an odd part $\lambda_i = 2t+1$ is replaced by $t+1$, t and an even part $\lambda_i = 2t$ by t, t. The resulting partition $([(\lambda_1+1)/2], [\lambda_1/2], [(\lambda_2+1)/2], \cdots, [\lambda_m/2])$ of n is denoted by $\mathrm{dbl}(\lambda)$, the doubling of λ.

EXAMPLES (3.5). (i) Doubling a $\bar{4}$-core $(4k+1, \cdots, 5, 1)$, respectively $(4k+3, \cdots, 7, 3)$, gives a 2-core κ_{2k+1}, respectively κ_{2k+2}.

(ii) For $\lambda = (13, 11, 8, 5, 2)$ we obtain $\mathrm{dbl}(\lambda) = (7, 6^2, 5, 4^2, 3, 2, 1^2)$.

LEMMA (3.6). *For $\lambda \in \mathscr{D}(n)$ we have*

$$\mathrm{dbl}(\lambda_{(\bar{4})}) = \mathrm{dbl}(\lambda)_{(2)}.$$

Hence, using Example 3.5(i) *above, the bar partitions with a fixed* $\bar{4}$-*core are exactly the bar partitions whose doublings have the corresponding* 2-*core.*

Proof. Sliding a bead up one position on the 0/2-runner, say $2k+2 \to 2k$, corresponds to removing a 2-hook from the corresponding doubling via $(k+1)^2 \to k^2$. So taking out all even parts of λ, the resulting partition has a doubling with the same 2-core as $\mathrm{dbl}(\lambda)$. If there are beads in position 1 and 3, then the doubling ends on $2, 1^2$, and this vanishes on removing two 2-hooks. Now consider a move $4k+5 \to 4k+1$. If there is no bead in position $4k+3$, then the doubling has parts $(\cdots, 2k+3, 2k+2, \ell, \ldots)$ with $\ell \leqslant 2k$ and we can remove two 2-hooks to obtain $(\cdots, 2k+1, 2k, \ell, \ldots)$. If there is a bead $4k+3$, the doubling has parts $(\cdots, 2k+3, (2k+2)^2, 2k+1, \ldots)$, and we can first remove two 2-hooks to obtain $(\cdots (2k+1)^4 \cdots)$, and then these parts all vanish by taking out further 2-hooks corresponding to taking out the beads $4k+1$, $4k+3$ when pushed to the top. The procedure is similar for the moves $4k+7 \to 4k+3$. So parallel to reducing the configuration of λ to that of $\lambda_{(\bar{4})}$, we reduce $\mathrm{dbl}(\lambda)$ to $\mathrm{dbl}(\lambda)_{(2)}$.

There is also another way of seeing the property above, by introducing an appropriate analogue of the 2-residue diagram and comparing the contents of λ and $\mathrm{dbl}(\lambda)$.

DEFINITION (3.7). Let $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathscr{D}(n)$, then its *shifted Young diagram* $S(\lambda)$ is obtained from the usual Young diagram by indenting the

ith row $(i-1)$ positions. The jth node in the ith row is called the (i, j)-node. The $\bar{4}$-residue of the (i, j)-node of $S(\lambda)$ is defined to be

$$\begin{array}{lll} 0 & \text{if } j-i\equiv 0 \text{ or } 3 & \pmod 4 \\ 1 & \text{if } j-i\equiv 1 \text{ or } 2 & \pmod 4 \end{array}$$

The $\bar{4}$-residue diagram of λ is obtained from $S(\lambda)$ by replacing the (i, j)-node by its $\bar{4}$-residue.

EXAMPLE. $\lambda=(13, 11, 8, 5, 2)\in\mathscr{D}(39)$ has $\bar{4}$-residue diagram

$$\begin{array}{ccccccccccccc} 0&1&1&0&0&1&1&0&0&1&1&0&0\\ &0&1&1&0&0&1&1&0&0&1&1&\\ &&0&1&1&0&0&1&1&0&&&\\ &&&0&1&1&0&0&&&&&\\ &&&&0&1&&&&&&& \end{array}$$

As before, we define a δ-invariant and content of a bar partition:

DEFINITION (3.8). Let $\lambda=(\lambda_1, \cdots, \lambda_m)\in\mathscr{D}(n)$. We set

$$\begin{aligned}\bar{\delta}(\lambda) = {} & \#\{0\text{'s in the } \bar{4}\text{-residue diagram of } \lambda\}\\ & -\#\{1\text{'s in the } \bar{4}\text{-residue diagram of } \lambda\}\end{aligned}$$

and define the $\bar{4}$-content of λ as

$$\begin{aligned}\bar{c}(\lambda) = (& \#\{0\text{'s in the } \bar{4}\text{-residue diagram of } \lambda\},\\ & \#\{1\text{'s in the } \bar{4}\text{-residue diagram of } \lambda\}).\end{aligned}$$

Clearly, removing an even part of λ or decreasing an odd part by 4 or removing parts 1 and 3 simultaneously does not change the $\bar{\delta}$-invariant, so

$$\bar{\delta}(\lambda)=\bar{\delta}(\lambda_{(\bar{4})}).$$

In fact, there is a good connection between the residue diagrams for λ and $\mathrm{dbl}(\lambda)$:

LEMMA (3.9). *For* $\lambda\in\mathscr{D}(n)$, $\bar{c}(\lambda)=c(\mathrm{dbl}(\lambda))$.

Proof. Taking a part of λ, the doubling means that the row in the $\bar{4}$-residue diagram is "zig-zagged" into two rows of the 2-residue diagram of $\mathrm{dbl}(\lambda)$:

$$\begin{array}{ccccccccccccccccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & \cdots \rightarrow & 0 & 1 & 0 & 1 & 0 & \\ & & & & & & & & & & & & & & & & \cdots, \\ & & & & & & & & & & & 1 & 0 & 1 & 0 & 1 & \end{array}$$

so the content is not changed.

In particular, this also implies that $\mathrm{dbl}(\lambda)_{(2)} = \mathrm{dbl}(\lambda_{(\bar{4})})$. Using the above and Proposition 3.1 we also deduce immediately:

PROPOSITION (3.10). *Define* $\mathscr{D}_n^k = \{\lambda \in \mathscr{D}(n) \mid \mathrm{dbl}(\lambda)_{(2)} = (k, k-1, ..., 1)\}$ *for* $k \in \mathbb{N}_0$, *and take* $\lambda \in \mathscr{D}_n^k$. *Then*

(a) *If* $\lambda \subset \mu \in \mathscr{D}(n+1)$, *then*

$$\mu \in \begin{cases} \mathscr{D}_{n+1}^{k+2} \cap \mathscr{D}_{n+1}^{k-2} & \textit{if} \quad k \geqslant 2 \\ \mathscr{D}_{n+1}^{0} \cap \mathscr{D}_{n+1}^{3} & \textit{if} \quad k = 1 \\ \mathscr{D}_{n+1}^{1} \cap \mathscr{D}_{n+1}^{2} & \textit{if} \quad k = 0 \end{cases}$$

(b) *If* $\rho \subset \lambda$, $\rho \in \mathscr{D}(n-1)$, *then*

$$\rho \in \begin{cases} \mathscr{D}_{n-1}^{k+2} \cap \mathscr{D}_{n-1}^{k-2} & \textit{if} \quad k \geqslant 2 \\ \mathscr{D}_{n-1}^{0} \cap \mathscr{D}_{n-1}^{3} & \textit{if} \quad k = 1 \\ \mathscr{D}_{n-1}^{1} \cap \mathscr{D}_{n-1}^{2} & \textit{if} \quad k = 0. \end{cases}$$

Finally, for the later study of decomposition numbers, we also need the concept of ladders in the $\bar{4}$-residue diagram.

The ladders in the $\bar{4}$-residue diagram are indicated by the lines joining the 0's and 1's:

$$\begin{array}{cccccccccccc} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \cdots \\ & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & & \\ & & 0 & 1 & 1 & 0 & 0 & 1 & 1 & & & \\ & & & & & & \ddots & & & & & \end{array}$$

More precisely, the 0-ladders connect the nodes (from bottom to top):

$$L_{i0}\colon (i, 1) \rightarrow (i-1, 5) \rightarrow (i-1, 4) \rightarrow (i-2, 9) \rightarrow (i-2, 8) \rightarrow \cdots \\ \rightarrow (1, 4(i-1)+1) \rightarrow (1, 4(i-1))$$

and the 1-ladders connect the nodes (from bottom to top):

$$L_{i1}\colon (i,3)\to(i,2)\to(i-1,7)\to(i-1,6)\to\cdots$$
$$\to(1,4(i-1)+3)\to(1,4(i-1)+2).$$

The ladders in a partition $\lambda\in\mathcal{D}(n)$ are then the intersections of these L_{ij} with (the $\bar{4}$-residue diagram of) λ, denoted by $L_{ij}(\lambda)$.

Note that in this situation it is not always possible to "regularize" a bar partition by pushing the beads to the top of the ladders:

EXAMPLE. $\lambda=(4,2,1)$

$$\begin{array}{cccc} 0 & 1 & 1 & 0 \\ & 0 & 1 & \\ & & 0 & \end{array} \xrightarrow{\text{pushing up}} \begin{array}{cccccccc} 0 & 1 & 1 & 0 & 0 & 1 & \cdot & 0, \end{array}$$

so we do *not* obtain the diagram of a partition.

But, in fact, for our purposes we do not want to "regularize" λ but rather its doubling.

From the proof of Lemma 3.9, the "zig-zagging" procedure shows that the 0- and the 1-ladders in the $\bar{4}$-residue diagram of $\lambda\in\mathcal{D}(n)$ are "stretched" into 0- and 1-ladders in the 2-residue diagram of $\operatorname{dbl}(\lambda)\in\mathcal{P}(n)$. So λ and $\operatorname{dbl}(\lambda)$ have the same distribution of beads into their respective ladders. We will see later that $\operatorname{dbl}^2(\lambda):=\operatorname{dbl}(\lambda)^R$ plays the same role for the decomposition numbers of the spin character $\langle\lambda\rangle$ as α^R plays for the character $[\alpha]$: it marks the final non-zero entry of this row in the decomposition matrix.

4. THE SPIN CHARACTERS IN A 2-BLOCK

The 2-blocks of S_n (and of $\hat{S}_n$) are determined by the 2-cores of the partitions of n. We have already noted before that such a non-empty 2-core has the form $\kappa_k=(k,k-1,\cdots,1)$, where $\binom{k+1}{2}\equiv n \bmod 2$. For emphasis, we denote in this section by $B^k_{n,w}$ the block of S_n of weight $w=(n-\binom{k+1}{2})/2$ with core $\kappa_k=(k,k-1,\cdots,1)$, so $n=\binom{k+1}{2}+2w$. When n is known we sometimes omit n and the weight and denote the block simply by B^k. The block of $\hat{S}_n$ containing $B^k_{n,w}$(resp. B^k) will be denoted by $\hat{B}^k_{n,w}$(resp. $\hat{B}^k$).

THEOREM (4.1). *Let $\lambda\in\mathcal{D}(n)$. Let $\langle\lambda\rangle$ be any spin character labeled by λ. Then $\langle\lambda\rangle\in\hat{B}^k_{n,w}$ if and only if* $\operatorname{dbl}(\lambda)$ *has* 2-*core* κ_k.

Special cases of this theorem have been known before, [Ben] for λ spin regular and [Ol2] for $w=0, 1$.

As mentioned in [Ol2] there is an easy way to determine for a given $\lambda \in \mathscr{D}(n)$ the core number k s.t. $\mathrm{dbl}(\lambda)_{(2)} = \kappa_k$:

LEMMA (4.2). *Suppose that* $\lambda \in \mathscr{D}(n)$ *has* q *parts* $\equiv 1 \pmod 4$ *and* r *parts* $\equiv 3 \pmod 4$. *Then* $\mathrm{dbl}(\lambda)_{(2)} = \kappa_k$ *where*

$$k = \begin{cases} 2(q-r)-1 & \text{if} \quad q > r \\ 2(r-q) & \text{if} \quad r \geqslant q. \end{cases}$$

In particular, the even parts of λ *do not play a role for* $\mathrm{dbl}(\lambda)_{(2)}$.

EXAMPLE. $\lambda = (7, 4, 3)$. Here $q=0$, $r=2$, $\mathrm{dbl}(\lambda)_{(2)} = (4, 3, 2, 1)$.

LEMMA (4.3). *Suppose that* $\lambda \in \mathscr{D}(n)$, $\langle \lambda \rangle \in \hat{B}^k_{n,w}$.

(1) *If* $\langle \mu \rangle$ *is a constituent of* $\langle \lambda \rangle \uparrow \hat{S}_{n+1}$, *then*

$$\begin{array}{llll} \langle \mu \rangle \in \hat{B}^{k-2}_{n+1, w+k} & \text{or} & \hat{B}^{k+2}_{n+1, w-(k+1)} & \text{when} \quad k \geqslant 2 \\ \langle \mu \rangle \in \hat{B}^{0}_{n+1, w+1} & \text{or} & \hat{B}^{3}_{n+1, w-2} & \text{when} \quad k = 1 \\ \langle \mu \rangle \in \hat{B}^{1}_{n+1, w} & \text{or} & \hat{B}^{2}_{n+1, w-1} & \text{when} \quad k = 0. \end{array}$$

(2) *If* $\langle \rho \rangle$ *is a constituent of* $\langle \lambda \rangle \downarrow \hat{S}_{n-1}$, *then*

$$\begin{array}{llll} \langle \rho \rangle \in \hat{B}^{k-2}_{n-1, w+k-1} & \text{or} & \hat{B}^{k+2}_{n-1, w-(k+2)} & \text{when} \quad k \geqslant 2 \\ \langle \rho \rangle \in \hat{B}^{0}_{n-1, w} & \text{or} & \hat{B}^{3}_{n-1, w-3} & \text{when} \quad k = 1 \\ \langle \rho \rangle \in \hat{B}^{1}_{n-1, w-1} & \text{or} & \hat{B}^{2}_{n-1, w-2} & \text{when} \quad k = 0. \end{array}$$

Proof. A similar statement has been stated in Section 3 for ordinary and modular characters in $B^k_{n,w}$. The statement must then also hold for the spin characters in $\hat{B}^k_{n,w}$ since $B^k_{n,w}$ and $\hat{B}^k_{n,w}$ have the same modular characters.

In the final steps of our inductive proof of (4.1) we will need the following rather special result:

LEMMA (4.4). (1) *If* $\ell > k \geqslant 0$ *then* $\langle 4\ell, 4\ell-1, \cdots, 4k+2 \rangle$ *is not in the principal block.*

(2) *If* $\ell \geqslant k \geqslant 0$ *then* $\langle 4\ell+3, 4\ell+2, \cdots, 4k+3 \rangle$ *is not in the principal block.*

Proof. Let $\hat{x} \in \hat{S}_n$ where $\pi(\hat{x}) = x$ is a 3-cycle in S_n. For computing the central character value $w_\chi(\hat{x}) = |S_n: C_{S_n}(x)| \, (\chi(\hat{x})/\chi(1))$, $\chi = \langle \lambda \rangle$, we use the analogue of the Murnaghan–Nakayama formula [Mo],

$$\langle \lambda \rangle (\hat{x}) = \sum_{H \text{ 3-bar}} (-1)^{\ell(H)} \, 2^{m(H)} \langle \lambda \backslash H \rangle (1),$$

where

$$m(H) = \begin{cases} 1 & \text{if} \quad \varepsilon(\lambda \backslash H) - \varepsilon(\lambda) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $\ell(H)$ is the leg length of the 3-bar H.

We then use the degree formula

$$\langle \lambda \rangle (1) = 2^{[(|\lambda| - \ell(\lambda))/2]} \frac{|\lambda|!}{\prod (\lambda_i !)} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}$$

for $\lambda = (\lambda_1, \lambda_2, \ldots)$.

First we deal with the case $\chi = \langle 4\ell, 4\ell - 1, \cdots, 4k + 2 \rangle$, $\ell > k \geqslant 0$.

If $k = 0$, we can remove only two 3-bars from $\lambda = (4\ell, 4\ell - 1, \cdots, 3, 2)$ and obtain $\lambda^1 = (4\ell, 4\ell - 1, \cdots, 4, 2)$, respectively $\lambda^2 = (4\ell, 4\ell - 1, \cdots, 5, 3, 2, 1)$. Note that $\varepsilon(\lambda^1) = 0$ and $\varepsilon(\lambda^2) = 1$, so $m(H^1) = 0$ and $m(H^2) = 1$ for the corresponding bars. For $\ell = 1$, we immediately obtain for $\chi = \langle 4, 3, 2 \rangle$:

$$w_\chi(\hat{x}) \equiv 1 \qquad (\text{mod } 2).$$

So we may assume $\ell > 1$. Then

$$w_\chi(\hat{x}) \equiv 2^{-1} \left(3! \prod_{i=4}^{4\ell} \frac{i+3}{i-3} + \frac{4!(\prod_{i=4}^{4\ell-1} i) \cdot 2(\prod_{i=9}^{4\ell+4} i) \, 7 \cdot 6}{(\prod_{i=6}^{4\ell+1} i) \cdot 4 \cdot 3(\prod_{i=1}^{4\ell-4} i) \cdot 2} \right) \qquad (\text{mod } 2)$$

where we have used for the first summand that we have lost only part 3 in going from λ to λ^1, and thus could cancel most of the terms in $\langle \lambda^1 \rangle (1) / \langle \lambda \rangle (1)$ coming from the degree formula above, except those terms for λ that involve the part 3. Similarly, in the second summand we are left with only the terms coming from λ that involve 4, respectively, those terms from λ^2 that involve the new part 1. Now we can further simplify:

$$\begin{aligned} w_\chi(\hat{x}) &\equiv \frac{(4\ell+2)(4\ell)(4\ell-2)}{2 \cdot 4 \cdot 6} + \frac{2 \cdot (4\ell-2)(4\ell+4)(4\ell+2)}{2 \cdot 8 \cdot 2} && (\text{mod } 2) \\ &\equiv \frac{2^4 \ell}{2^4} + \frac{2^5(\ell+1)}{2^5} && (\text{mod } 2) \\ &\equiv 2\ell + 1 \qquad \equiv 1 && (\text{mod } 2). \end{aligned}$$

Next we turn to the case $k>0$. Here we can strip off three different 3-bars and obtain the partitions

$$\lambda^1 = (4\ell, \cdots, 4k+3, 4k-1)$$
$$\lambda^2 = (4\ell, \cdots, 4k+4, 4k+2, 4k)$$
$$\lambda^3 = (4\ell, \cdots, 4k+5, 4k+3, 4k+2, 4k+1)$$

which are all odd, hence $m(H)=1$ in all three cases.

In case $\ell = k+1$, all the partitions λ, λ^1, λ^2, λ^3 have three parts, and we leave it to the reader to check that $w_\chi(\hat{x}) \equiv 1 \pmod 2$. So we now assume $\ell > k+1$. With similar cancellation as above we obtain:

$$\begin{aligned}
w_\chi(\hat{x}) &\equiv \left(2^3 k \frac{\prod_{i=4}^{4\ell-4k+1} i \prod_{i=8k+3}^{4\ell+4k+2} i}{\prod_{i=8k+2}^{4\ell+4k-1} i \prod_{i=1}^{4\ell-4k-2} i} + 2 \frac{\prod_{i=2}^{4\ell-4k} i \prod_{i=8k+7}^{4\ell+4k+3} i}{\prod_{i=8k+2}^{4\ell+4k} i \prod_{i=1}^{4\ell-4k-3} i} \right. \\
&\quad \left. + 2^3(k+1) \frac{\prod_{i=1}^{4\ell-4k-1} i \prod_{i=8k+9}^{4\ell+4k+4} i \cdot 2}{\prod_{i=8k+3}^{4\ell+4k+1} i \prod_{i=1}^{4\ell-4k-4} i \cdot 2} \right) \\
&\equiv 2^2 k \frac{4(\ell-k) \cdot 2 \cdot 4(\ell+k)}{2 \cdot 2} + \frac{4(\ell-k) \cdot 2 \cdot 2}{2 \cdot 4 \cdot 2} \\
&\quad + 2^2(k+1) \frac{2 \cdot 4(\ell+k+1) \cdot 2}{4 \cdot 2 \cdot 8(k+1)} \\
&\equiv 2^5 k(\ell-k)(\ell+k) + (\ell-k) + (\ell+k+1) \equiv 1 \pmod 2.
\end{aligned}$$

So in all cases $w_\chi(\hat{x}) \not\equiv 0 \pmod 2$, and hence χ is not in the principal block.

Next we have to deal with the character χ labeled by

$$\lambda = (4\ell+3, 4\ell+2, \cdots, 4k+3), \qquad \ell \geqslant k \geqslant 0.$$

For $\ell = k$ one easily checks:

$$w_\chi(\hat{x}) \equiv 2^{[(4k-1)/2]+1+[(4k+2)/2]} \frac{(4k+3)!}{(4k)!} \equiv 1 \pmod 2.$$

For $\ell > k$, calculations similar to those in the previous case show that again always $w_\chi(\hat{x}) \not\equiv 0 \pmod 2$. Hence $\langle 4\ell+4, ..., 4k+3 \rangle$ also is not in the principal block.

We are now ready to start the proof of (4.1). It is by induction on n and then for a given n by induction on the (relevant) weights of blocks.

Using the character tables of spin characters for "small" n, (4.1) is easily verified directly in these cases. Suppose that (4.1) has been verified for $\hat{S}_1, ..., \hat{S}_{n-1}$. Let $\lambda \in \mathscr{D}(n)$ (or $\mathscr{D}(n+1)$). We are going to say that λ is *known*, if it can be verified that $\langle \lambda \rangle$ is in the correct block using only the induction hypothesis, i.e., the validity of (4.1) for 1, ..., $n-1$. We are also going to use the following abbreviating notation for λ:

$$\lambda=(a_1, b_1; a_2, b_2; \ldots; a_t, b_t) \quad \text{where} \quad a_1 \geqslant b_1 > a_2+1 > b_2 > a_3+1 > \cdots > b_t > 0.$$

This is to signify that

$$\lambda=(a_1, a_1-1, \cdots, b_1, a_2, a_2-1, \cdots, b_2, \ldots, a_t, a_t-1, \cdots, b_t).$$

We call b_1, ..., b_t the *smallest parts* and a_1, ..., a_t the *largest parts* of λ. Thus for

$$\lambda=(9, 8, 7, 6, 4, 2, 1) \qquad (a_1=9, b_1=6, a_2=b_2=4, a_3=2, b_3=1)$$

we write

$$\lambda=(9, 6; 4, 4; 2, 1).$$

For $1 \leqslant i, j \leqslant t$, λ^i is the partition obtained from λ by replacing the part a_i by the part a_i+1; λ_j is the partition obtained from λ by replacing the part b_j of λ by the part b_j-1, and λ^i_j is the partition obtained from λ by replacing the parts a_i and b_j by a_i+1 and b_j-1. (Note that λ^i_j is *not defined* in the following cases: $i=j$, $a_i=b_i$, $j=i-1$ and $b_j=a_i+2$.)

Trivially,

$$|\lambda|=|\lambda^i_j|, \qquad |\lambda^i|=|\lambda|+1, \qquad |\lambda_j|=|\lambda|-1.$$

We call

$$\begin{tikzcd}
& \lambda^i \arrow[dr, dash] & \\
\lambda \arrow[ur, dash] \arrow[dr, dash] & & \lambda^i_j \\
& \lambda_j \arrow[ur, dash] &
\end{tikzcd}$$

the (i, j)-*diagram* of λ.

We keep the notation above for the rest of this section.

LEMMA (4.5). *If $\lambda \in \mathscr{D}^-(n)$, then λ is known.*

Proof. Let $1 \leqslant j \leqslant t$. By the induction hypothesis λ_j is known, say $\langle \lambda_j \rangle \in B^k$. Since by the Branching Theorem $\langle \lambda \rangle$ is a constituent of $\langle \lambda_j \rangle \uparrow \hat{S}_n$ we get by (4.3) that $\langle \lambda \rangle$ is in one of two possible blocks. Note that the weight of these two blocks have *different parities*. But by (2.19) the parity of the weight of the block containing $\langle\lambda\rangle$ is determined by (the 2-singular weight of) λ. Thus only one possibility remains for the block containing $\langle\lambda\rangle$. The result follows from (3.10).

In the remainder of the proof we consider only $\lambda \in \mathscr{D}^+(n)$.

We call two integers a and b *related* if they are both congruent to 0 or 1 (mod 4) or both congruent to 2 or 3 (mod 4). Thus e.g. 13 and 16 are related, but 13 and 14 are not.

LEMMA (4.6). *If $\lambda \in \mathscr{D}(n)$ and, for some i and j, b_i and b_j are not related, then λ is known.*

Proof. Since b_i and b_j are not related, λ_i and λ_j have different contents. Thus the constituents $\langle\lambda_i\rangle$ and $\langle\lambda_j\rangle$ of $\langle\lambda\rangle_{|\hat{S}_{n-1}}$ are in different 2-blocks of $\hat{S}_{n-1}$, by the inductive hypothesis. But $\langle\lambda\rangle$ is a constituent of $\langle\lambda_i\rangle \uparrow \hat{S}_n$ and of $\langle\lambda_j\rangle \uparrow \hat{S}_n$, and (4.3) shows that there is only one possibility for the block of $\langle\lambda\rangle$. The result follows from (3.10).

LEMMA (4.7). *Assume that $\lambda \in \mathscr{D}^+(n)$ and that all the b_i's are related. If for some i, $a_i + 1$ is related to b_i, then λ is known.*

Proof. For all j, $\mathrm{dbl}(\lambda)^i$ and $\mathrm{dbl}(\lambda)_j$ have different 2-cores. Indeed, a node with the same residue is added to λ to get λ^i as is removed from λ to get λ_j. Therefore $\delta(\mathrm{dbl}(\lambda)^i) \neq \delta(\mathrm{dbl}(\lambda)_j)$. Consider the (i, j)-diagrams

$$\begin{tikzcd}[column sep=small] & \lambda^i \arrow[dl, dash] \arrow[dr, dash] & \\ \lambda \arrow[dr, dash] & & \lambda^i_j \\ & \lambda_j \arrow[ur, dash] & \end{tikzcd}$$

for all possible j. We claim that if λ^i_j *is known for some j then λ is known.* Indeed, if λ^i_j is known, then the argument of (4.5) shows that λ^i is known: We know that $\lambda^i \in \mathscr{D}^-(n+1)$, because $\lambda \in \mathscr{D}^+(n)$ (λ and λ^i have the same number of parts!). By the induction hypothesis λ_j is also known and thus $\langle\lambda^i\rangle$ and $\langle\lambda_j\rangle$ are in blocks with different cores. Then by (4.3) there is only one possibility for the block of $\langle\lambda\rangle$ (the same as for $\langle\lambda^i_j\rangle$). Therefore by (3.10) λ is known.

We may therefore assume that λ^i_j may not be determined using (4.6) for any relevant j. The assumptions imply that λ has more than one part. If $t = 1 (= i)$ then $b_1 + 1$ and $b_1 - 1$ are smallest parts of λ^1_1 which are not related and λ_1^1 is determined unless $b_1 = 1$. In this case 2 and $a_1 + 1$ are non-related smallest parts of λ_1^1, and λ_1^1 is determined.

We assume $t > 1$. Assume also $t > i$.

If b_t is even, then $b_t - 1$ is not related to the other b_j's and λ_t^i is known. Thus b_t is odd. Unless $a_t = b_t$, $b_t + 1$ is a smallest part not related to the b_j's and λ_t^i is known. Thus $a_t = b_t$ is odd. A similar argument repeated for $t-1, t-2, \ldots$ shows:

$$\text{For } t > j > i, \qquad b_j = a_j \quad \text{is odd.}$$

However, b_i is not odd since this would force $a_i = b_i$ to be odd, con-
tradic-
ting the assumption that $a_i + 1$ is related to a_i. Thus b_i is even. We get $a_i = b_i$, since otherwise $b_i - 1$ is a smallest part of λ_i^i. It is not possible that $b_i - 1 = a_{i+1} + 1$ since a_{i+1} is odd if it occurs at all. Since $\lambda \in \mathscr{D}^+(n)$, $i \neq 1$. But in both cases, whether b_{i-1} is odd or even, it is possible by the above arguments to know λ_{i-1}^i and we are done.

We have now reached the stage where all $\lambda \in \mathscr{D}(n)$ are known by the inductive hypothesis unless:

(i) $\lambda \in \mathscr{D}^+$,

(ii) all b_j's are related,

(iii) all $a_i + 1$'s are related,

(iv) no b_j is related to an $a_i + 1$.

We call a λ satisfying (i)–(iv) a $*$*-partition.* If $\lambda = (a_1, b_1; a_2, b_2, \cdots, a_t, b_t)$ is a $*$-partition then indeed all $\mathrm{dbl}(\lambda)^i$ and all $\mathrm{dbl}(\lambda)_j$ have the *same* 2-core and it is possible to describe their 2-core (and that of $\mathrm{dbl}(\lambda)$) explicitly. We call a pair (a_i, b_i) occurring in λ a *sequence* in λ.

LEMMA (4.8). *Assume that λ is a $*$-partition.*

(1) *If the b_i's are related to* 1 *then the sequences in λ have the form*

$$(a_i, b_i) = \begin{cases} (4\ell + 1, 4k) & (\mathscr{D}^-) \\ (4\ell + 2, 4k) & (\mathscr{D}^+) \\ (4\ell + 2, 4k + 1) & (\mathscr{D}^-) \\ (4\ell + 1, 4k + 1) & (\mathscr{D}^+) \end{cases}$$

for suitable $k, \ell \in \mathbb{N} \cup \{0\}$.

(2) *If the b_j's are related to* 3, *then the sequences in* λ *have the form*

$$(a_i, b_i) = \begin{cases} (4\ell + 3, 4k + 2) & (\mathscr{D}^-) \\ (4\ell, 4k + 2) & (\mathscr{D}^+) \\ (4\ell, 4k + 3) & (\mathscr{D}^-) \\ (4\ell + 3, 4k + 3) & (\mathscr{D}^+) \end{cases}$$

for suitable $k, \ell \in \mathbb{N} \cup \{0\}$.

The sign $\mathscr{D}^+/\mathscr{D}^-$ *following the sequence signifies whether there is an even or an odd number of even integers in the sequence* $a_i, a_i - 1, ..., b_i$.

Proof. The proof is immediate from the definition of a $*$-partition.

LEMMA (4.9). *Let* λ *be a* $*$ *-partition.*

(1) *If the* b_i*'s are related to* 1, *then the* 2*-core of* $\mathrm{dbl}(\lambda)$ *is* $(2t-1, 2t-2, \cdots, 1) = \kappa_{2t-1}$, *and the* 2*-core of each* $\mathrm{dbl}(\lambda)_j$ *is* κ_{2t-3} *when* $t \geqslant 2$ *and* $\varnothing = \kappa_0$ *if* $t = 1$. *Thus*

$$w_2(\mathrm{dbl}(\lambda)_j) - w_2(\mathrm{dbl}(\lambda)) = 2t - 2.$$

(2) *If the* b_i*'s related to* 3, *then the* 2*-core of* $\mathrm{dbl}(\lambda)$ *is* $(2t, 2t-1, \cdots, 1) = \kappa_{2t}$ *and the* 2*-core of each* $\mathrm{dbl}(\lambda)_j$ *is* κ_{2t-2} *when* $t \geqslant 2$ *and* $\varnothing = \kappa_0$ *if* $t = 1$.

Proof. (1) By (4.8) each of the t sequences $a_i, a_i - 1, ..., b_i$ contains exactly one odd part congruent 1 (mod 4) *more* than odd parts congruent to 3 (mod 4). The claim about $\mathrm{dbl}(\lambda)$ follows from (4.2). In λ_j the sequences $\neq j$ are still present. However, the (parity) change $b_j \to b_j - 1$ means that $a_j, a_j - 1, ..., b_j + 1, b_j - 1$ give no contribution to the 2-core of $\mathrm{dbl}(\lambda)_j$, again by (4.2).

(2) is proved similarly.

The statement about the weight is an easy calculation.

For the rest of the proof we adopt the following notation. Let $\lambda \in \mathscr{D}(n)$. We define $\hat{w}(\lambda)$ to be the 2-weight of $\mathrm{dbl}(\lambda)$. If $\langle \lambda \rangle \in \hat{B}_{n,w}$ we define $w(\lambda) = w$. To conclude the proof of our theorem we show:

LEMMA 4.10. *For all* $\lambda \in \mathscr{D}(n)$, $\hat{w}(\lambda) \leqslant w(\lambda)$, *except possibly when* n *is odd,* λ *is a* $*$*-partition with* $t = 1$, *and* b_1 *is related to* 1. *In this case* $\hat{w}(\lambda) = (n-1)/2$ *and* $w(\lambda) \geqslant (n-3)/2$.

Proof. If λ is not a $*$-partition we have the equality $\hat{w}(\lambda)=w(\lambda)$ by (4.5)–(4.7). Consider a $*$-partition $\lambda=(a_1, b_1, a_2, b_2;\ldots; a_t, b_t)$. By (4.9) and the induction hypothesis

$$\langle\lambda_1\rangle \in \begin{cases} \hat{B}^{2t-3} & \text{if} \quad t \geqslant 2 \\ \hat{B}^0 & \text{if} \quad t=1 \end{cases} \qquad \text{if } b_1 \text{ is related to 1}$$

$$\langle\lambda_1\rangle \in \hat{B}^{2t-2} \qquad \text{if } b_1 \text{ is related to 3}$$

Therefore, by (4.3)

$$\langle\lambda\rangle \in \begin{cases} \hat{B}^{2t-1} & \text{or} \quad \hat{B}^{2t-5} & \text{if} \quad t \geqslant 3 \\ \hat{B}^3 & \text{or} \quad \hat{B}^0 & \text{if} \quad t=2 \\ \hat{B}^1 & \text{or} \quad \hat{B}^2 & \text{if} \quad t=1 \end{cases} \qquad \text{if } b_1 \text{ is related to 1}$$

$$\langle\lambda\rangle \in \begin{cases} \hat{B}^{2t} & \text{or} \quad \hat{B}^{2t-1} & \text{if} \quad t \geqslant 2 \\ \hat{B}^1 & \text{or} \quad \hat{B}^2 & \text{if} \quad t=1 \end{cases} \qquad \text{if } b_1 \text{ is related to 3}$$

Moreover, by (4.9)

$$\mathrm{dbl}(\lambda)_{(2)}=\begin{cases} \kappa_{2t-1} & \text{if} \quad t>1 \\ \kappa_0=\varnothing & \text{if} \quad t=1 \end{cases} \qquad \text{if } b_1 \text{ is related to 1}$$

$$\mathrm{dbl}(\lambda)_{(2)}=\kappa_{2t} \qquad \text{if } b_1 \text{ is related to 3}$$

Thus we get that the inequality $\hat{w}(\lambda) \leqslant w(\lambda)$ holds for all $*$-partitions except possibly for $t=1$ and b_1 related to 1. In this case $|\lambda|$ is odd.

Conclusion of the Proof of Theorem 4.1. We assume that the result is true in $\hat{S}_1, \ldots, \hat{S}_{n-1}$. For any block $\hat{B}_{n,w}$ of $\hat{S}_n$ we know that $\#\{\lambda \in \mathscr{D}^+(n) \mid \hat{w}(\lambda)=w\}$ (respectively $\#\{\lambda \in \mathscr{D}^-(n) \mid \hat{w}(\lambda)=w\}$) equals the number of self-associate (resp. pairs of non-self-associate) spin characters in $\hat{B}_{n,w}$. This follows from the results of Sections 2 and 3. Therefore, if we know for a given $w_1 \in \mathbb{N}$ that

$$\text{For all } \lambda \in \mathscr{D}(n),\ w(\lambda) \leqslant w_1 \Rightarrow \hat{w}(\lambda) \leqslant w_1, \tag{$*$}$$

then Theorem 4.1 is true for all blocks in $\hat{S}_n$ of weight $\leqslant w_1$. This is seen by induction on the relevant weights for the blocks of $\hat{S}_n$, which are less than or equal to w_1. Therefore Lemma 4.10 shows that Theorem 4.1 is true if n is even or if n is odd and $w_1<(n-3)/2$. For the blocks $\hat{B}^1$ and $\hat{B}^2$ of weight $(n-1)/2$ and $(n-3)/2$ of $\hat{S}_n$, n odd, all spin characters in the block are determined except those of the form $\langle\lambda\rangle$, λ a $*$-partition with $t=1$. But then by Lemma 4.4, $\langle\lambda\rangle$, λ a $*$-partition with $t=1$ and b_1 related to 3, is *not* in $\hat{B}^1$. Therefore it is in $\hat{B}^3$. This means that since we know the number of spin characters in $\hat{B}^3$, we have determined all spin characters in $\hat{B}^3$. Therefore, the remaining spin characters $\langle\lambda\rangle$, λ a $*$-partition with $t=1$ and b_1 related to 1, are in $\hat{B}^1$, and we are done.

5. ON THE DECOMPOSITION MATRIX FOR $\hat{S}_n$ IN CHARACTERISTIC 2

It is the aim of this section to obtain information on the decomposition matrix for $\hat{S}_n$ in characteristic 2 which is as precise on the rows corresponding to spin characters as it is for the rows corresponding to ordinary characters, where the information is given by James' results. The methods used are similar to the ones used in [BMO] in the characteristic 3 case. For $\lambda \succ n$ we set

$$\langle\hat{\lambda}\rangle = \begin{cases} \langle\lambda\rangle & \text{if } \lambda \text{ is even} \\ \langle\lambda\rangle + \langle\lambda\rangle' & \text{if } \lambda \text{ is odd.} \end{cases}$$

THEOREM (5.1). *For any 2-regular partition β of n there is a projective character*

$$\psi_\beta = \sum_{\substack{\alpha \vdash n \\ \alpha_{(2)} = \beta_{(2)}}} s_\alpha[\alpha] + \sum_{\substack{\lambda \succ n \\ \mathrm{dbl}(\lambda)_{(2)} = \beta_{(2)}}} t_\lambda \langle\hat{\lambda}\rangle$$

satisfying the following conditions:

(i) $s_\alpha \neq 0$ *only if* $\alpha^R \trianglelefteq \beta$ ($\alpha^R =$ *regularization of* $\alpha \vdash n$).

(ii) $s_\alpha = 1$ *if* $\alpha^R = \beta$.

(iii) $t_\lambda \neq 0$ *only if* $\mathrm{dbl}^2(\lambda) \trianglelefteq \beta$.

(iv) $t_\lambda = 2^{[m_0(\lambda)/2]}$ *if* $\mathrm{dbl}^2(\lambda) = \beta$, *where* $m_0(\lambda) = \#\{$*even parts of* $\lambda\}$.

In addition, $s_\alpha = 0$ (*resp*. $t_\lambda = 0$) *if* α (*resp*. λ) *cannot be "built along β-ladders"*.

Proof. Let β be a 2-regular partition of n. Consider the ladders of β in the 2-residue diagram and let $\beta_0 \subseteq \beta$ be the partition containing all the complete ladders of β. Then add ladders in β one at a time to obtain the sequence

$$\beta_0, \beta_1, \cdots, \beta_e = \beta$$

where β_i is a 2-regular partition of n_i, $n_e = n$. Set $k_i = n_i - n_{i-1}$ and let $r_i = 1$ or 0 if $\beta_i \backslash \beta_{i-1}$ consists of 1- or 0-nodes, respectively.

Now start at $\beta_0 = \kappa_k$. The ordinary character $[\beta_0]$ belongs to a block of $\hat{S}_{n_0}$ of weight 0, which only contains $[\beta_0]$ and the spin character $\langle \lambda_0 \rangle$ (see [Ol2]),

$$\lambda_0 = \begin{cases} (2k-1, 2k-5, ..., 5, 1) & \text{if } \ k \text{ is odd} \\ (2k-1, 2k-5, ..., 7, 3) & \text{if } \ k \text{ is even.} \end{cases}$$

It also contains only one 2-modular irreducible $\phi(\beta_0)$.

One checks $\dim\langle \lambda_0 \rangle = \dim[\beta_0] = \dim \phi(\beta_0)$ (or uses [Ben]) to obtain the decomposition matrix for a block of weight 0,

	$\phi(\beta_0)$
$[\beta_0]$	1
$\langle \lambda_0 \rangle$	1

i.e. we have a projective character $\psi_{\beta_0} = [\beta_0] + \langle \lambda_0 \rangle$. As $\mathrm{dbl}(\lambda_0) = \beta_0$, the conditions of the theorem are satisfied.

Thus we can now assume $e > 0$. By [BMO, Proposition 2.4] we obtain the projective character

$$\psi_{\beta_0} \uparrow^{S_{n_1}}_{(r_1)} = k_1! \left(\sum_{\beta_0 \xrightarrow[(r_1)]{k_1} \alpha} [\alpha] + \sum_{\lambda_0 \xrightarrow[(r_2)]{k_1} \lambda} 2^{a(\lambda, \lambda_0)} \langle \hat{\lambda} \rangle \right),$$

where

$$a(\lambda, \lambda_0) = \frac{1}{2} (i(\lambda \backslash \lambda_0) - d(\lambda \backslash \lambda_0) + \varepsilon(\lambda_0) - \varepsilon(\lambda))$$

$$= \frac{1}{2} (m_0(\lambda) - \varepsilon(\lambda)) = \left\lfloor \frac{m_0(\lambda)}{2} \right\rfloor.$$

Since all ladders in λ_0 and β_0 are full, we can add k_1 r_1-nodes only by putting them all into the next ladder, and thus $\beta_0 \xrightarrow[(r_1)]{k_1} \alpha$ implies $\alpha^R = \beta_1$ and $\lambda_0 \xrightarrow[(r_1)]{k_1} \lambda$ implies $\mathrm{dbl}^2(\lambda) = \beta_1$.

Hence dividing by $k_1!$ we obtain a projective character

$$\psi_{\beta_1} = \sum_{\alpha : \alpha^R = \beta_1} [\alpha] + \sum_{\substack{\lambda \succ n_1 : \\ \mathrm{dbl}^2(\lambda) = \beta_1}} 2^{[m_0(\lambda)/2]} \langle \hat{\lambda} \rangle.$$

If $e = 1$, we are done; otherwise we continue:

Again by [BMO, Proposition 2.4] and r_2-inducing we now obtain a projective character

$$\psi_{\beta_1}\uparrow^{S_{n_2}}_{(r_2)} = k_2!\left(\sum_{\alpha:\,\alpha^R=\beta_1}\ \sum_{\alpha \xrightarrow[(r_2)]{k_2} \varphi} [\varphi] + \sum_{\lambda:\,\mathrm{dbl}^2(\lambda)=\beta_1} 2^{[m_0(\lambda)/2]} \sum_{\lambda \xrightarrow[(r_2)]{k_2} \mu} 2^{a(\mu,\lambda)}\langle\hat{\mu}\rangle\right).$$

As the r_2-ladders of α with $\alpha^R=\beta_1$ and of λ with $\mathrm{dbl}^2(\lambda)=\beta_1$ are all complete, any φ with $\alpha \xrightarrow[(r_2)]{k_2} \varphi$ (resp., any μ with $\lambda \xrightarrow[(r_2)]{k_2} \mu$) satisfies $\varphi^R=\beta_2$ (resp. $\mathrm{dbl}^2(\mu)=\beta_2$). Furthermore, any such φ (resp. μ) comes from a unique α (resp. λ) by stripping off the outermost r_2-ladder of φ (resp. μ). Hence,

$$\psi_{\beta_1}\uparrow^{S_{n_2}}_{(r_2)} = k_2!\left(\sum_{\varphi:\,\varphi^R=\beta_2} [\varphi] + \sum_{\mathrm{dbl}^2(\mu)=\beta_2} 2^{[m_0(\lambda)/2]+a(\mu,\lambda)}\langle\hat{\mu}\rangle\right).$$

Now

$$\begin{aligned}\left[\frac{m_0(\lambda)}{2}\right] + a(\mu,\lambda) &= \frac{1}{2}\left(m_0(\lambda) - \varepsilon(\lambda) + i(\mu\backslash\lambda) - d(\mu\backslash\lambda) + \varepsilon(\mu)\right)\\ &= \frac{1}{2}\left(\underbrace{m_0(\lambda) + (i(\mu\backslash\lambda) - d(\mu\backslash\lambda))}_{m_0(\mu)} - \varepsilon(\mu)\right)\\ &= \left[\frac{m_0(\mu)}{2}\right],\end{aligned}$$

so by dividing out $k_2!$ we have a projective character

$$\psi_{\beta_2} = \sum_{\varphi:\,\varphi^R=\beta_2} [\varphi] + \sum_{\mu:\,\mathrm{dbl}^2(\mu)=\beta_2} 2^{[m_0(\mu)/2]}\langle\hat{\mu}\rangle.$$

We continue now by induction on e, the result (in a slightly stronger form) proved so far for $e \leqslant 2$.

By the induction hypothesis we have a projective character

$$\begin{aligned}\psi_{\beta_{e-1}} = &\sum_{\alpha:\,\alpha^R=\beta_{e-1}} [\alpha] + \sum_{\lambda:\,\mathrm{dbl}^2(\lambda)=\beta_{e-1}} 2^{[m_0(\lambda)/2]}\langle\hat{\lambda}\rangle\\ &+ \sum_{\alpha:\,\alpha^R \lhd \beta_{e-1}} s_\alpha[\alpha] + \sum_{\lambda:\,\mathrm{dbl}^2(\lambda)\lhd\beta_{e-1}} t_\lambda\langle\hat{\lambda}\rangle.\end{aligned}$$

By r_e-induction we construct a projective character

$$\psi_{\beta_{e-1}}\uparrow^{S_{n_e}}_{(r_e)}=k_e!\Bigg(\sum_{\varphi:\,\varphi^R=\beta_e}[\varphi]+\sum_{\substack{\mu:\,\mathrm{dbl}^2(\mu)=\beta_e\\ \lambda_\mu=\mu\backslash L_\mu}}2^{[m_0(\lambda_\mu)/2]+a(\mu,\lambda_\mu)}\langle\hat{\mu}\rangle$$
$$+\text{“lower terms”}\Bigg)$$

where we have already used the following facts:

(i) $\alpha\xrightarrow[(r_e)]{k_e}\varphi$, $\alpha^R\trianglelefteq\beta_{e-1}\Rightarrow\varphi^R\trianglelefteq\beta_e$

(ii) $\varphi^R=\beta_e\Rightarrow\exists$ unique α with $\alpha^R\trianglelefteq\beta_{e-1}$ and $\alpha\xrightarrow[(r_e)]{k_e}\varphi$, namely $\alpha=\varphi\backslash L_\varphi$, where L_φ denotes the outermost r_e-ladder of φ

(iii) $\lambda\xrightarrow[(r_e)]{k_e}\mu$, $\mathrm{dbl}^2(\lambda)\trianglelefteq\beta_{e-1}\Rightarrow\mathrm{dbl}^2(\mu)\trianglelefteq\beta_e$

(iv) $\mathrm{dbl}^2(\mu)=\beta_e\Rightarrow\exists$ unique λ with $\mathrm{dbl}^2(\lambda)\trianglelefteq\beta_{e-1}$ and $\lambda\xrightarrow[(r_e)]{k_e}\mu$, namely $\lambda=\lambda_\mu=\mu\backslash L_\mu$.

Here (iii) and (iv) come from the observation that the ladders of λ (resp. μ) in the shifted 4-diagram are just “stretched” to obtain the ladders of $\mathrm{dbl}^2(\lambda)$ (resp. $\mathrm{dbl}^2(\mu)$) in the 2-residue diagram.

Now

$$\begin{aligned}\left[\frac{m_0(\lambda_\mu)}{2}\right]+a(\mu,\lambda_\mu)&=\frac{1}{2}(m_0(\lambda_\mu)-\varepsilon(\lambda_\mu)+i(\mu\backslash\lambda_\mu)-d(\mu\backslash\lambda_\mu)+\varepsilon(\lambda_\mu)-\varepsilon(\mu))\\&=\frac{1}{2}(\underbrace{m_0(\lambda_\mu)+i(\mu\backslash\lambda_\mu)-d(\mu\backslash\lambda_\mu)}_{m_0(\mu)}-\varepsilon(\mu))\\&\qquad\text{(because }\lambda_\mu=\mu\backslash L_\mu)\\&=\left[\frac{m_0(\mu)}{2}\right]\end{aligned}$$

Hence, by dividing out $k_e!$ we obtain a projective character

$$\psi_{\beta_e}=\sum_{\varphi:\,\varphi^R=\beta_e}[\varphi]+\sum_{\mu:\,\mathrm{dbl}^2(\mu)=\beta_e}2^{[m_0(\mu)/2]}\langle\hat{\mu}\rangle+\text{“lower terms”}$$

as desired.

The additional statement in the theorem is obvious from the above inductive proof.

The theorem already gives a rather good approximation for the decomposition matrix. Order the column labels lexicographically, decreasing from left to right, and the rows as follows: first the ordinary characters, the ones labeled by 2-regular partitions coming first, these in lexicographical ordering, decreasing from top to bottom, then the rest, also in lexicographical order; then the spin characters in lexicographical ordering, with only one of a pair of associate characters as these have the same 2-modular decomposition. Note that dominance order implies lexicographical order, thus our theorem gives the following result on the shape of the decomposition matrix ("reduced" according to the above convention of taking only one of a pair of associate spin characters):

THEOREM (5.2). *The 2-modular decomposition matrix of $\hat{S}_n$ has the following shape w.r.t. the ordering described above*:

$$
\begin{array}{c|cccccc c}
 & \phi(\beta_1) & \cdots & \phi(\beta_j) & \cdots & \cdots & \phi(\beta_l) & \\
\hline
[\alpha_1] & 1 & & & 0 & & & \\
\vdots & & \ddots & & & & & \\
 & * & & & & & & \\
[\alpha_l] & & & & \ddots & & & \\
 & & & & & & 1 & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
[\alpha_{l+1}] & & & & * & & & \\
\vdots & & & & & & & \\
[\alpha_i] & \cdots & & 1 & 0 & \cdots & 0 & \alpha_i^R = \beta_j \\
\vdots & & & & & & & \\
[\alpha_m] & & & * & & & & \\
\hline
\langle \lambda_1 \rangle & & \vdots & & & * & & \\
\vdots & & & & & & & \\
 & * & & & & & & \\
\langle \lambda_k \rangle & \cdots & & 2^{\left[\frac{m_0(\lambda_k)}{2}\right]} & 0 & \cdots & 0 & \mathrm{dbl}^2(\lambda_k) = \beta_j \\
\vdots & & & & & & & \\
\langle \lambda_t \rangle & & & * & & & &
\end{array}
$$

Proof. Given the approximation matrix w.r.t. the ordering described above, we obtain the projective indecomposables by subtracting from the right only, because of the upper unitriangular part. In a row corresponding to a spin character $\langle \lambda_k \rangle$, $2^{[m_0(\lambda_k)/2]}$ is the final non-zero entry in the approximation matrix, in the column labeled by $\mathrm{dbl}^2(\lambda_k)$, hence also in the decomposition matrix. (The argument is similar for a row corresponding to an ordinary character, but this is well-known.)

Remark. For "spin regular" bar partitions, i.e., $\lambda = (\lambda_1 > \cdots > \lambda_m)$ with the property that

$$\left(\left[\frac{\lambda_1+2}{2}\right], \left[\frac{\lambda_1-1}{2}\right], \left[\frac{\lambda_2+2}{2}\right], \ldots, \left[\frac{\lambda_m-1}{2}\right]\right)$$

is a 2-regular partition (which then equals $\mathrm{dbl}^2(\lambda)$), Benson [Ben] has proved the result on the corresponding row of the decomposition matrix with very different methods. The idea that this should generalize to arbitrary bar partitions λ as proved above, i.e. (using his notation):

$$\overline{\langle \lambda \rangle} \sim 2^{[m_0(\lambda)/2]} D^{\mathrm{dbl}^2(\lambda)} + \sum_{\beta \rhd \mathrm{dbl}^2(\lambda)} c_\beta D^\beta,$$

was one of the motives of our investigation.

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